Cohomology of antiPoisson superalgebra

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Abstract

We consider antiPoisson superalgebras realized on the smooth Grassmann-valued functions with compact support in $\mathbb{R}^n$ and with the grading inverse to Grassmanian parity. The lower cohomologies of these superalgebras are found.

1 Introduction

The odd Poisson bracket play an important role in Lagrangian formulation of the quantum theory of the gauge fields, which is known as BV-formalism [1], [2] (see also [3]-[4]). In [6], it was shown that there are two analogs of the Poisson bracket and related "mechanics": a direct one, still called the Poisson bracket, and the "odd" one, introduced in physical literature in [1] under the name "antibracket".

The antibracket possesses many features analogous to those of the Poisson bracket and even can be obtained via "canonical formalism" with an "odd time". However, unlike the Poisson bracket, on different aspects of whose deformations (quantization) there is voluminous literature, the deformations of the antibracket is not satisfactorily studied yet. The only result is [7], where the deformations of the Poisson and antibracket superalgebras realized on the superspace of *polynomials* are found.

The goal of present work is finding the lower cohomology spaces of antiPoisson superalgebra realized on the *smooth* Grassmann-valued functions with compact support in $\mathbb{R}^n$. These results is used in the next work [8] where the general form of the deformation of such antiPoisson superalgebra is found. Particularly, it is shown in [8] that the nontrivial deformations do exist.

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of smooth $\mathbb{K}$-valued functions with compact support on $\mathbb{R}^n$. This space is endowed with its standard topology. We set

$$\mathbf{D}_{n_+}^{n_-} = \mathcal{D}(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad \mathbf{E}_{n_+}^{n_-} = C^\infty(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad \mathbf{D}_{n_+}'^{n_-} = \mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-},$$

where $\mathbb{G}^{n_-}$ is the Grassmann algebra with n_- generators and $\mathcal{D}'(\mathbb{R}^{n_+})$ is the space of continuous linear functionals on $\mathcal{D}(\mathbb{R}^{n_+})$. The generators of the Grassmann algebra (resp., the coordinates of the space $\mathbb{R}^{n_+}$) are denoted by ξ^α for $\alpha = 1, \ldots, n_-$ (resp., x^i for $i = 1, \ldots, n_+$). We shall also use common notation z^A which are equal to x^A for $A = 1, \ldots, n_+$ and to ξ^{A-n_+} for $A = n_+ + 1, \ldots, n_+ + n_-$.

The spaces $\mathbf{D}_{n_+}^{n_-}$, $\mathbf{E}_{n_+}^{n_-}$, and $\mathbf{D}_{n_+}'^{n_-}$ possess a natural parity which is determined by that of the Grassmann algebra, it is denoted by ε. Set: $\epsilon = \varepsilon + 1$.

We set $\varepsilon_A = 0$ for $A = 1, \ldots, n_+$ and $\varepsilon_A = 1$ for $A = n_+ + 1, \ldots, n_+ + n_-$.

It is well known, that if $n_+ = n_- = n$ then the bracket

$$[f,g](z) = \sum_{i=1}^n \left(f(z) \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\partial}{\partial \xi^i} g(z) - f(z) \frac{\overleftarrow{\partial}}{\partial \xi^i} \frac{\partial}{\partial x^i} g(z) \right), \tag{1.1}$$

called *antibracket*, defines a Lie superalgebra structure on the superspaces $\mathbf{D}_n \overset{def}{=} \mathbf{D}_n^n$ and $\mathbf{E}_n \overset{def}{=} \mathbf{E}_n^n$ with the ϵ-parity. Clearly, the form ω defining the antibracket

$$[f,g](z) = f(z) \frac{\overleftarrow{\partial}}{\partial z^A} \omega^{AB} \frac{\partial}{\partial z^B} g(z),$$

is constant, non-degenerate, and satisfies the condition

$$\omega^{BA} = -(-1)^{\epsilon_A \epsilon_B} \omega^{AB}, \; \epsilon(\omega^{AB}) = \epsilon_A + \epsilon_B,$$

Here these Lie superalgebras are called the *antiPoisson superalgebras*. [1] We set: $\mathbf{D}_n' \overset{def}{=} \mathbf{D}_n'^n$.

The integral on $\mathbf{D}_n$ is defined by the relation

$$\int dz\, f(z) = \int_{\mathbb{R}^n} dx \int d\xi\, f(z),$$

where the integral on the Grassmann algebra is normalized by the condition

$$\int d\xi\, \xi^1 \ldots \xi^n = 1.$$

We identify $\mathbb{G}^n$ with its dual space $\mathbb{G}'^n$ setting

$$f(g) = \int d\xi\, f(\xi) g(\xi) \text{ for any } f, g \in \mathbb{G}^n.$$

Accordingly, the space $\mathbf{D}_n'$ of continuous linear functionals on $\mathbf{D}_n$ is identified with $\mathcal{D}'(\mathbb{R}^n) \otimes \mathbb{G}^n$. The value $m(f)$ of a functional $m \in \mathbf{D}_n'$ on a test function $f \in \mathbf{D}_n$ will be often written in the integral form:

$$m(f) = \int dz\, m(z) f(z).$$

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[1] We will also consider the usual multiplication of the elements of the antiPoisson superalgebras with commutation relations $fg = (-1)^{\varepsilon(f)\varepsilon(g)} gf$, so the x^i will be called even variables and the ξ^i will be the odd ones.

2 Cohomology of the antiPoisson superalgebras (Results)

Let $\mathbf{D}_n$ act in a $\mathbb{Z}_2$-graded space V (the action of $f \in \mathbf{D}_n$ on $v \in V$ will be denoted by $f \cdot v$). The space $C^p(\mathbf{D}_n, V)$ of p-cochains consists, as in [10], of all separately continuous multilinear superantisymmetric maps $\mathbf{D}_n^p \longrightarrow V$. Superantisymmetry means, as usual, that

$$M_p(\dots, f_i, f_{i+1}, \dots) = -(-1)^{\epsilon(f_i)\epsilon(f_{i+1})} M_p(\dots, f_{i+1}, f_i, \dots).$$

The space $C^p(\mathbf{D}_n, V)$ possesses a natural $\mathbb{Z}_2$-grading:

$$\epsilon(M_p(f_1, \dots, f_p)) = \epsilon_{M_p} + \epsilon(f_1) + \dots + \epsilon(f_p)$$

for any (homogeneous) $f_j \in \mathbf{D}_n$. We will often use the Grassmann ε-parity[2] of cochains: $\varepsilon_{M_p} = \epsilon_{M_p} + p + 1$. The differential $d_p^V : C^p(\mathbf{D}_n, V) \longrightarrow C^{p+1}(\mathbf{D}_n, V)$ is defined as follows:

$$d_p^V M_p(f_1, \dots, f_{p+1}) = -\sum_{j=1}^{p+1}(-1)^{j+\epsilon(f_j)|\epsilon(f)|_{1,j-1}+\epsilon(f_j)\epsilon_{M_p}} f_j \cdot M_p(f_1, \dots, \breve{f}_j, \dots, f_{p+1}) -$$

$$-\sum_{i<j}(-1)^{j+\epsilon(f_j)|\epsilon(f)|_{i+1,j-1}} M_p(f_1, \dots f_{i-1}, [f_i, f_j], f_{i+1}, \dots, \breve{f}_j, \dots, f_{p+1}), \tag{2.1}$$

for any $M_p \in C^p(\mathbf{D}_n, V)$ and $f_1, \dots, f_{p+1} \in \mathbf{D}_n$ having definite ϵ-parities. Here the sign $\breve{}$ means that the argument is omitted and

$$|\epsilon(f)|_{i,j} = \sum_{l=i}^{j} \epsilon(f_l).$$

We have $d_{p+1}^V d_p^V = 0$ for any $p = 0, 1, \dots$. The p-th cohomology space of the differential d_p^V will be denoted by H_V^p. The second cohomology space H_{ad}^2 in the adjoint representation is closely related to computing infinitesimal deformations of the Lie bracket $[\cdot, \cdot]$ in the form

$$[f, g]_* = [f, g] + \hbar[f, g]_1 + \dots$$

up to similarity transformations

$$[f, g]_T = T^{-1}[Tf, Tg],$$

where a continuous linear operator $T : V[[\hbar^2]] \longrightarrow V[[\hbar^2]]$ is of the form $T = \mathsf{id} + \hbar^2 T_1$. The condition that $[\cdot, \cdot]_1$ is a 2-cocycle is equivalent to the Jacobi identity for $[\cdot, \cdot]_*$ modulo the $\hbar^2$-order terms.

We study the cohomology of the antiPoisson superalgebra $\mathbf{D}_n$ in the following cases:

1. The trivial representation: $V = \mathbb{K}$, $f \cdot a = 0$ for any $f \in \mathbf{D}_n$ and $a \in \mathbb{K}$. (Notation: H_{tr}^p and d_p^{tr}).

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[2]If V is the space of Grassmann-valued functions on $\mathbb{R}^n$, then ε defined in such a way coincides with the usual Grassmann parity.

2. $V = \mathbf{D}'_n$ and $f \cdot g = [f, g]$ for any $f \in \mathbf{D}_n$, $g \in \mathbf{D}'_n$. (Notation: $H^p_{\mathbf{D}'}$ and d^{ad}_p).

3. $V = \mathbf{E}_n$ and $f \cdot g = [f, g]$ for any $f \in \mathbf{D}_n$, $g \in \mathbf{E}_n$. (Notation: $H^p_{\mathbf{E}}$ and d^{ad}_p).

4. The adjoint representation: $V = \mathbf{D}_n$ and $f \cdot g = [f, g]$ for any $f, g \in \mathbf{D}_n$. (Notation: H^p_{ad} and d^{ad}_p).

In the case of the trivial representation, $\mathbb{K}$ is considered as a superspace whose ϵ-even subspace is zero. We say that the p-cocycles $M^1_p, \ldots M^k_p$ are *independent* if they give rise to linearly independent elements in H^p. For a multilinear form M_p taking values in $\mathbf{D}_n$, $\mathbf{E}_n$, or $\mathbf{D}'_n$, we write $M_p(z|f_1, \ldots, f_p)$ instead of more cumbersome $M_p(f_1, \ldots, f_p)(z)$.

The following theorems describe lower cohomology of the antiPoisson superalgebra.

Theorem 2.1.

1. $H^1_{\mathrm{tr}} \simeq 0$.

2. Let $n \geq 2$. Then $H^2_{\mathrm{tr}} \simeq 0$.

   Let $n = 1$. Then $H^2_{\mathrm{tr}} \simeq \mathbb{K}^2$, and the cochains

$$\mu_1(f, g) = \int dz(-1)^{\varepsilon(f)}\{\partial^3_x \partial_\xi f(z)\}g(z), \quad \mu_2(f, g) = \int dz(-1)^{\varepsilon(f)}\{\partial^2_x f(z)\}g(z) \quad (2.2)$$

   are independent nontrivial cocycles.

It follows from Theorem 2.1 that if $n = 1$, then the antiPoisson superalgebra has a 2-parametric central extension. These extensions are described in [8].

Now, let $\mathbf{Z}_n = \mathbf{D}_n \oplus \mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n)$, where $\mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n)$ is a centralizer of $\mathbf{D}_n$ in $\mathbf{E}_n$. Clearly, $\mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n) = \mathbb{K}$.

Theorem 2.2.

1. $H^0_{\mathbf{D}'} \simeq H^0_{\mathbf{E}} \simeq \mathbb{K}$; the function $m_0(z) \equiv 1$ is a nontrivial cocycle.

   $H^0_{\mathrm{ad}} \simeq 0$.

2. (a) $H^1_{\mathbf{D}'} \simeq H^1_{\mathbf{E}} \simeq \mathbb{K}^2$; independent nontrivial cocycles are given by

$$m_{1|1}(z|f) = \mathcal{E}_z f(z), \quad m_{1|2}(z|f) = \Delta f(z),$$

   where

$$\mathcal{E}_z = 1 - \frac{1}{2}z^A \frac{\partial}{\partial z^A}, \quad \Delta = \frac{1}{2}(-1)^{\varepsilon_A}\omega^{AB}\frac{\partial}{\partial z^A}\frac{\partial}{\partial z^B} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi^i}. \quad (2.3)$$

   (b) Let V_2 be the two-dimensional subspace of $C^1(\mathbf{D}_n, \mathbf{D}_n)$ generated by the cocycles $m_{1|1}$ and $m_{1|2}$. Then there is a natural isomorphism $V_2 \oplus (\mathbf{E}_n/\mathbf{Z}_n) \simeq H^1_{\mathrm{ad}}$ taking $(M_1, T) \in V_2 \oplus (\mathbf{E}_n/\mathbf{Z}_n)$ to the cohomology class determined by the cocycle $M_1(z|f) + [t(z), f(z)]$, where $t \in \mathbf{E}_n$ belongs to the equivalence class T.

3. *Let the bilinear maps $m_{2|1}$, $m_{2|2} : (\mathbf{D}_1)^2 \longrightarrow \mathbf{E}_1$ and $m_{2|3}$ $m_{2|4} : (\mathbf{D}_n)^2 \longrightarrow \mathbf{E}_n$ be defined by the relations*

$$m_{2|1}(z|f,g) = \int du \partial_\eta g(u) \partial_y^3 f(u), \quad \epsilon_{m_{2|1}} = 1, \tag{2.4}$$

$$m_{2|2}(z|f,g) = \int du \theta(x-y)[\partial_\eta g(u) \partial_y^3 f(u) - \partial_\eta f(u) \partial_y^3 g(u)] +$$

$$+ x[\{\partial_\xi \partial_x^2 f(z)\}\partial_\xi \partial_x g(z) - \{\partial_\xi \partial_x f(z)\}\partial_\xi \partial_x^2 g(z)], \quad \epsilon_{m_{2|2}} = 1, \tag{2.5}$$

$$m_{2|3}(z|f,g) = (-1)^{\varepsilon(f)}\{(1-N_\xi)f(z)\}(1-N_\xi)g(z), \quad \epsilon_{m_{2|3}} = 1, \tag{2.6}$$

$$m_{2|4}(z|f,g) = (-1)^{\varepsilon(f)}\{\Delta f(z)\}\mathcal{E}_z g(z) + \{\mathcal{E}_z f(z)\}\Delta g(z) \quad \epsilon_{m_{2|4}} = 0. \tag{2.7}$$

where $z = (x, \xi)$, $u = (y, \eta)$, $N_\xi = \xi \partial_\xi$.

Then

(a) *$H_{\mathrm{ad}}^2 \simeq \mathbb{K}^2$ and the cochains $m_{2|3}(z|f,g)$ and $m_{2|4}(z|f,g)$ are independent nontrivial cocycles.*

(b) *Let $n = 1$.*
   Then $H_{\mathbf{D}'}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^4$ and the cochains $m_{2|1}(z|f,g)$, $m_{2|2}(z|f,g)$, $m_{2|3}(z|f,g)$, and $m_{2|4}(z|f,g)$ are independent nontrivial cocycles.

(c) *Let $n \geq 2$. Then $H_{\mathbf{D}'}^2 \simeq H_{\mathbf{E}}^2 \simeq \mathbb{K}^2$ and the cochains $m_{2|3}(z|f,g)$ and $m_{2|4}(z|f,g)$ are independent nontrivial cocycles.*

3   Preliminary and Notation

We define δ-function by the formula

$$\int dz' \delta(z'-z)f(z') = \int f(z')\delta(z-z')dz' = f(z).$$

Evidently,

$$[f,g](z) = (-1)^{\varepsilon_A \epsilon(f)}\frac{\partial}{\partial z^A}(f(z)\omega^{AB}\frac{\partial}{\partial z^B}g(z)) - 2f\Delta g(z),$$

$$(-1)^{\epsilon(g)}\int dz f[g,h] = \int dz[f,g]h + 2\int dz f \Delta gh,$$

where Δ is defined by (2.3).

The following notation is used below:

$$T_{...(A)_k...} \equiv T_{...A_1...A_k...}, \quad T_{...A_i A_{i+1}...} = (-1)^{\varepsilon_{A_i}\varepsilon_{A_{i+1}}}T_{...A_{i+1}A_i...}, \quad i = 1, \ldots, k-1$$

$$T_{...(A)_k...}Q_{...}{}^{(A)_k}{}_{...} \equiv T_{...A_1...A_k...}Q_{...}{}^{A_1...A_k}{}_{...},$$

$$(\partial_A)^Q \equiv \partial_{A_1}\partial_{A_2}\ldots\partial_{A_Q}, \quad (p_A)^Q \equiv p_{A_1}p_{A_2}\ldots p_{A_Q},$$

and so on.

We denote by $M_p(\ldots)$ the separately continuous superantisymmetrical p-linear forms on $(\mathbf{D}_n)^p$. Thus, the arguments of these functionals are the functions $f(z)$ of the form

$$f(z) = \sum_{k=0}^{n} f_{(\alpha)_k}(x)(\xi^\alpha)^k \in \mathbf{D}_n, \quad f_{(\alpha)_k}(x) \in \mathcal{D}(\mathbb{R}^n). \tag{3.1}$$

For any $f(z) \in \mathbf{D}_n$ we can define the support

$$\mathrm{supp}(f) \stackrel{def}{=} \bigcup_{(\alpha)_k} \mathrm{supp}(f_{(\alpha)_k}(x)).$$

For each set $V \subset \mathbb{R}^n$ we use the notation $z \bigcap V = \varnothing$ if $z = (x, \xi)$ and there exist some domain $U \subset \mathbb{R}^n$ such that $x \in U$ and $U \bigcap V = \varnothing$.

It can be easily proved that such multilinear forms can be written in the integral form (see [10]):

$$M_p(f_1, \ldots, f_p) = \int dz_p \cdots dz_1 m_p(z_1, \ldots, z_p) f_1(z_1) \cdots f_p(z_p), \ p = 1, 2, \ldots \tag{3.2}$$

and

$$M_p(z|f_1, \ldots, f_p) = \int dz_p \cdots dz_1 m_p(z|z_1, \ldots, z_p) f_1(z_1) \cdots f_p(z_p), \ p = 1, 2, \ldots . \tag{3.3}$$

Let by definition

$$\epsilon(M_p(f_1, \ldots, f_p)) = \epsilon_{m_p} + pn + \epsilon(f_1) + \ldots + \epsilon(f_p).$$

It follows from the properties of the forms M_p that the corresponding kernels m_p have the following properties:

$$\epsilon_{m_p} = pn + \epsilon_{M_p}, \quad \varepsilon_{m_p} = pn + \varepsilon_{M_p}, \quad \epsilon_{m_p} = \varepsilon_{m_p} + p + 1,$$
$$m_p(*|z_1 \ldots z_i, z_{i+1} \ldots z_p) = (-1)^n m_p(*|z_1 \ldots z_{i+1}^*, z_i^* \ldots z_p). \tag{3.4}$$

Here $z^* = (x, -\xi)$ if $z = (x, \xi)$.

Introduce the space $\mathcal{M}_1 \subset C^2(\mathbf{D}_n, \mathbf{D}_n')$ consisting of all 2-forms which can be locally represented as

$$M_{2|2}^1(z|f, g) = \sum_{q=0}^{Q} m^{1(A)_q}(z|[(\partial_A^z)^q f(z)]g - (-1)^{\epsilon(f)\epsilon(g)}[(\partial_A^z)^q g(z)]f), \tag{3.5}$$

with locally constant Q and the space $\mathcal{M}_2 \subset C^2(\mathbf{D}_n, \mathbf{D}_n')$ consisting of all 2-forms which can be locally represented as

$$M_{2|2}^2(z|f, g) = \sum_{q=0}^{Q} m^{2(A)_q}(z|[(\partial_A)^q f]g - (-1)^{\epsilon(f)\epsilon(g)}(\partial_A)^q g]f) \tag{3.6}$$

with locally constant Q, where $m^{1,2(A)_q}(z|\cdot) \in C^1(\mathbf{D}_n, \mathbf{D}_n')$.

The space $\mathcal{M}_0 = \mathcal{M}_1 \bigcap \mathcal{M}_2$ is called in this paper the space of local bilinear forms. It consists of all the form, which can be present as

$$M_{2|\,\mathrm{loc}}(z|f,g) = \sum_{p,\,q=0}^{Q} m^{(A)_q|(B)_p}(z)\left((\partial_A)^q f(z)\,(\partial_B)^p g(z) - (-1)^{\epsilon(f)\epsilon(g)}(\partial_A)^q g(z)\,(\partial_B)^p f(z)\right).$$

Here $m^{(A)_q|(B)_p} \in D' \otimes \mathbb{G}^n$, and the limit Q is locally constant with respect to z.

The following low degree filtrations $\mathcal{P}_p$ and $\mathcal{P}_{p,q}$ of the polynomials we will use in what follows:

Definition.
$$\mathcal{P}_p = \{f(k) \in \mathbf{E}_n[k] : \exists g \in \mathbf{E}_n[\alpha, k] \quad f(\alpha k) = \alpha^p g(\alpha, k)\},$$
$$\mathcal{P}_{p,q} = \{f(k_1, k_l) \in \mathbf{E}_n[k_1, k_2] : \exists g \in \mathbf{E}_n[\alpha, \beta, k_1, k_2] \quad f(\alpha k_1, \beta k_2) = \alpha^p \beta^q g(\alpha, \beta, k_1, k_2)\}.$$

Evidently, $\mathcal{P}_{p,q} \subset \mathcal{P}_{r,s}$ if $p \geq r$ and $q \geq s$. It is clear also, that if $f \in \mathcal{P}_{p,q}$ and $g \in \mathcal{P}_{r,s}$ then $fg \in \mathcal{P}_{p+r,q+s}$. Analogous relations are valid for $\mathcal{P}_p$.

4 Cohomologies in the trivial representation

In this section, we prove the theorem 2.1.

4.1 H^1_{tr}

Let $M_1(f) = \int dz m_1(z) f(z)$. Then the cohomology equation has the form $0 = d_1^{\mathrm{tr}} M_1(f,g) = -M_1([f,g]) = -\int dz m_1(z)[f(z), g(z)]$, and hence $m_1(z)\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}f(z) + 2m_1(z)\Delta f(z) = 0$. Finally, $m_1(z) = 0$, i.e., $H^1_{\mathrm{tr}} = 0$.

4.2 H^2_{tr}

For the bilinear form $M_2(f,g) = \int du dz m_2(z,u) f(z) g(u)$ the cohomology equation has the form

$$M_2([f,g],h) - (-1)^{\epsilon(g)\epsilon(h)} M_2([f,h],g) - M_2(f,[g,h]) = 0. \tag{4.1}$$

Let $\mathrm{supp}(h) \bigcap [\mathrm{supp}(f) \bigcup \mathrm{supp}(g)] = \varnothing$. Then we have $\hat{M}_2([f,g],h) = 0$ (the hat means that the corresponding form (or kernel) is considered out of the diagonal), which imply $\hat{m}_2(z,u) = 0$. Thus we can represent $M_2(f,g)$ in the following form (see [10])

$$M_2(f,g) = \sum_{k=0}^{K} \int dz m_2^{(A)_k}(z)\left((-1)^{\varepsilon(f)}(\partial_A)^k f(z) \cdot g(z) - (-1)^{\varepsilon(g)+\epsilon(f)\epsilon(g)}(\partial_A)^k g(z) \cdot f(z)\right),$$

$$\tag{4.2}$$

where upper limit K is locally constant.

Proposition 4.1. *Summation in formula (4.2) is made over even k.*

In particular, the highest degree K of derivatives in formula (4.2) is even, $K = 2m$.

Indeed, let K_0 be the highest odd degree of derivatives. Then a summands with $k = K_0$ in the second term is equal to

$$\int dz m_2^{(A)K_0}(z)(-1)^{\varepsilon(f)\varepsilon(g)}[(\partial_A)^{K_0}g(z)]f(z) =$$
$$= -\int dz m_2^{(A)K_0}(z)[(\partial_A)^{K_0}f(z)]g(z) + \sum_{k<K_0}\int dz w_2^{(A)_k}(z)g(z)(\partial_A)^k f(z)$$

and terms with $k = K_0$ are canceled in Exp. (4.2).

The cohomology equation has the form

$$\sum_{k=0}^{2m}\int dz m_2^{(A)_k}((-1)^{\varepsilon(f)+\varepsilon(g)}\{((\partial_A)^k[f,g])h + (-1)^{(\varepsilon(f)+\varepsilon(g)+1)\varepsilon(h)}((\partial_A)^k h)[f,g]\} +$$
$$+(-1)^{\varepsilon(g)\varepsilon(h)+\varepsilon(f)+\varepsilon(g)}\{((\partial_A)^k[f,h])g + (-1)^{\varepsilon(g)(\varepsilon(f)+\varepsilon(h)+1)}((\partial_A)^k g)[f,h]\} +$$
$$+(-1)^{\varepsilon(f)}\{((\partial_A)^k f)[g,h] + (-1)^{\varepsilon(f)[\varepsilon(g)+\varepsilon(h)+1]}((\partial_A)^k[g,h])f\}) = 0. \qquad (4.3)$$

Analogously to [9], take the functions in the form $f(z) \to e^{zp}f(z)$, $g(z) \to e^{zq}g(z)$, $h(z) \to e^{-z(p+q)}h(z)$, and consider the terms of the highest order in p and q which equals to $2m + 2$. Using the notation $\langle p, q \rangle \equiv \sum_{A,B}(-1)^{\varepsilon_A}\omega^{AB}p_A q_B = \langle q, p \rangle$ $(\varepsilon(\langle p, q \rangle) = 1)$ and, introducing the generation function

$$F_m(z, p) \equiv m_2^{(A)2m}(z)(p_A)^{2m},$$

we obtain

$$\langle p, p \rangle F_m(z, q) + \langle q, q \rangle F_m(z, p) + \langle p, q \rangle\{F_m(z, p) + F_m(z, q) - F_m(z, p+q)\} = 0 \qquad (4.4)$$

Let $p = (v_i, \theta_i)$, $q = (y_i, \zeta_i)$. Then we can rewrite (4.4) in the form

$$2v\theta F_m(z, q) + 2y\zeta F_m(z, p) + (x\zeta + \theta y)\{F_m(z, p) + F_m(z, q) - F_m(z, p+q)\} = 0 \qquad (4.5)$$

Proposition 4.2. *Let $n > 1$, $m \geq 2$. Then*

$$F_m(z, p) = 0. \qquad (4.6)$$

Proof. Let us differentiate Eq. (4.5) by $\partial_{q^B}\partial_{q^A}$ at $q = 0$. We obtain

$$\theta^i\partial_{v^j}F_m(z, p) + \theta^j\partial_{v^i}F_m(z, p) = 0 \implies \partial_{v^i}F_m(p) = 2\theta^i a_1(z, p),$$
$$v^i\partial_{\theta^j}F_m(z, p) - v^j\partial_{\theta^i}F_m(z, p) = 0 \implies \partial_{\theta^i}F_m(z, p) = 2v^i a_2(z, p),$$
$$2\delta_{ij}F_m(z, p) + \theta^i\partial_{\theta^j}F_m(z, p) - v^j\partial_{v^i}F_m(z, p) = 0 \implies$$

$$\delta_{ij}F_m(z, p) = v^j\theta^i a(z, p) = v^i\theta^j a(z, p), \quad a(z, p) = a_1(z, p) - a_2(z, p)$$

Let $i = 1$, $j = \bar{j} \neq 1$ $(j = (1, \bar{j}))$. We have

$$0 = v^{\bar{j}}\theta^1 a(z, p) = v^1\theta^{\bar{j}}a(z, p) \implies \theta^i a(z, p) = 0 \implies F_m(z, p) = 0.$$

Proposition 4.3. *Let $n > 1$, $m = 1$. Then*

$$F_1(z, p) = \frac{1}{2} b_1(z) \langle p, p \rangle, \qquad (4.7)$$

i.e., $F_1(z, p)$ corresponds to the differential of some 1-form (see Exp. (4.11)).
Proof.

$$F_1(z, p) = \frac{1}{2} p_A p_B P(z)_{AB}, \quad P_{AB} = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij}^t & c_{ij} \end{pmatrix}$$

From (4.5) it follows

$$2v\theta a_{ij} + \theta^i(a_{jk}v^k + b_{jk}\theta^k) + \theta^j(a_{ik}v^k + b_{ik}\theta^k) = 0 \implies b_{ik} = b_1\delta_{ik}, \quad a_{ij} = 0,$$
$$2v\theta c_{ij} + v^i(v^j + c_{jk}\theta^k) - v^j(v^i + c_{ik}\theta^k) = 0 \implies c_{ij} = 0 \implies P(z)_{AB} = b_1(-1)^{\varepsilon_A}\omega^{AB}.$$

Proposition 4.4. *Let $n > 1$, $m = 0$. Then $F_0(z, p) = 0$.*
Proof. Indeed, $F_0(z, p) = b_0(z)$, and it follows from (4.5) that

$$(2v\theta + 2y\zeta + v\zeta + \theta y)b_0(z) = 0 \implies b_0(z) = 0. \qquad (4.8)$$

Proposition 4.5. *Let $n = 1$. Then*

$$F(z, p) = F_{1|1}(z)v^2 + F_{2|1}(z)v\theta + F_{2|2}(z)v^3\theta.$$

Proof. We have in the case under consideration

$$F_m(z, p) = F_{1|m}(z)v^{2m} + F_{2|m}(z)v^{2m-1}\theta.$$

Eq. (4.5) takes the form

$$2v\theta(F_{1|m}y^{2m} + F_{2|m}y^{2m-1}\zeta) + 2y\zeta(F_{1|m}v^{2m} + F_{2|m}v^{2m-1}\theta) +$$
$$+ v\zeta(F_{1|m}v^{2m} + F_{2|m}v^{2m-1}\theta + F_{1|m}y^{2m} - F_{1|m}(v + y)^{2m} - F_{2|m}(v + y)^{2m-1}\theta) +$$
$$+ y\theta(F_{1|m}v^{2m} + F_{1|m}y^{2m} + F_{2|m}y^{2m-1}\zeta - F_{1|m}(v + y)^{2m} - F_{2|m}(v + y)^{2m-1}\zeta) = 0.$$

So

$$F_{1|m}[2vy^{2m} + v^{2m}y + y^{2m+1} - (v + y)^{2m}y] = 0, \qquad (4.9)$$
$$F_{2|m}[2vy^{2m-1} - 2v^{2m-1}y - v^{2m} + v(v + y)^{2m-1} + y^{2m} - (v + y)^{2m-1}y] = 0. \qquad (4.10)$$

Setting $y = v$ in Eq. (4.9), we obtain $(4 - 2^{2m})v^{2m+1}F_{1|m}(z) = 0$, such that $F_{1|m}(z) = 0$ if $m \neq 1$.

For $m \geq 3$ Eq. (4.10) takes the form $F_{2|m}(z)[2(m - 2)v^{2m-1}y + O(v^{2m-2})] = 0$ and so $F_{2|m}(z) = 0$ for $m \geq 3$. Eq. (4.9) for $m = 1$ and Eq. (4.10) for $m = 1, 2$ are identically satisfied for arbitrary $F_{1|1}(z)$, $F_{2|1}(z)$, and $F_{2|2}(z)$. ∎

Note that $F_{2|1}(z)v\theta$ corresponds to some coboundary, i.e. to the differential of some 1-form.

Return to the complete form of cohomology equation.

It is useful to rewrite a trivial solution of cohomology equation (4.1),

$$M_{2|\text{tr}}(f,g) = d_1^{\text{tr}} M_1(f,g) = \int dz\, m_1(z)[f(z), g(z)],$$

in the form (4.2),

$$M_{2|\text{tr}}(f,g) = -\int dz\, m_1(z)(-1)^{\varepsilon(f)} \left([\Delta f(z)]g(z) + (-1)^{\varepsilon(f)\varepsilon(g)}[\Delta g(z)]f(z)\right) -$$
$$- \int dz\, m_1(z)\overleftarrow{\Delta}(-1)^{\varepsilon(f)} f(z)g(z). \tag{4.11}$$

Consider the case $n > 1$. According to Eqs. (4.6), (4.7) and (4.11), $M_2(f,g)$ can be represented in the form

$$M_2(f,g) = d_1^{\text{tr}} M_1(f,g) + M_2'(f,g), \quad m_1(z) = -b_1(z),$$

$$M_2'(f,g) = \int dz\, m'^0(z)(-1)^{\varepsilon(f)} f(z)g(z), \quad m'^0(z) = m^0(z) + m_1(z)\overleftarrow{\Delta}.$$

The form $M_2'(f,g)$ satisfies the cohomology equation (4.1), and therefore, according to Eq. (4.8), $M_2'(f,g) = 0$. Thus, we have proved the following proposition

Proposition 4.6. *Let $n > 1$. Then the general solution of the cohomology equation (4.1) has the form*

$$M_2(f,g) = d_1^{\text{tr}} M_1(f,g).$$

Now consider the case $n = 1$. According to above consideration, in this case the solution of the cohomology equation (4.1) can be represented in the form

$$M_2(f,g) = M_{2|4}(f,g) + M_{2|2}(f,g) + M_{2|0}(f,g) + d_1^{\text{tr}} M_1(f,g),$$

where

$$M_{2|4}(f,g) = \int dz\, m_4(z)(-1)^{\varepsilon(f)} \{[\partial_x^3 \partial_\xi f(z)]g(z) + (-1)^{\varepsilon(f)\varepsilon(g)}[\partial_x^3 \partial_\xi g(z)]f(z)\},$$

$$M_{2|2}(f,g) = \int dz(-1)^{\varepsilon(f)} \{[m_{2|1}(z)\partial_x^2 f(z)]g(z) + (-1)^{\varepsilon(f)\varepsilon(g)}[m_{2|1}(z)\partial_x^2 g(z)]f(z)\},$$

$$M_{2|0}(f,g) = \int dz\, m_0(z)(-1)^{\varepsilon(f)} f(z)g(z), \quad M_1(f) = \int dz\, m_{2|2}(z)f(z).$$

Take the functions in the form $f(z) \to e^{zp}f(z)$, $g(z) \to e^{zq}g(z)$, $h(z) \to e^{-z(p+q)}h(z)$, and consider the terms of the sixth and fifth orders in p and q in cohomology equation (4.1). Only $M_{2|4}(f,g)$ from $M_2(f,g)$ will take part to such terms. The sixth order in p and q terms are cancel identically and we obtain the following equation for the fifth order terms

$$\partial_x m_4(z)(2v^3 + 3v^2y - 3vy^2 - 2y^3)\theta\zeta +$$
$$+ m_4(z)\overleftarrow{\partial}_\xi[(2v^3y + 3v^2y^2 + vy^3)\theta + (v^3y + 3v^2y^2 + 2vy^3)\zeta] = 0. \tag{4.12}$$

It follows from Eq. (4.12), that $\partial_x m_4(z) = m_4(z)\overleftarrow{\partial}_\xi = 0$. So $m_4(z) = \frac{1}{2}m_1 = \text{const.}$ Thus, we can write

$$M_2(f,g) = m_1\mu_1(f,g) + \bar{M}_2(f,g) + d_1^{\text{tr}}M_1(f,g), \quad \bar{M}_2(f,g) = M_{2|2}(f,g) + M_{2|0}(f,g),$$

$$\mu_1(f,g) = \int dz(-1)^{\varepsilon(f)}\{\partial_x^3\partial_\xi f(z)\}g(z), \quad \epsilon_{\mu_1} = 1. \tag{4.13}$$

The form $\mu_1(f,g)$ satisfies cohomology equation (4.1), such that the form $\bar{M}_2(f,g)$ satisfies cohomology equation (4.1) too. Again, take the functions in the form $f(z) \to e^{zp}f(z)$, $g(z) \to e^{zq}g(z)$, $h(z) \to e^{-z(p+q)}h(z)$, and consider the terms of the fourth and third orders in p and q in cohomology equation (4.1). Only $M_{2|2}(f,g)$ from $\bar{M}_2(f,g)$ will take part to such terms. The fourth order in p and q terms are cancelled identically and we obtain the equation for the third order terms

$$[\partial_x^2 m_{2|1}(z)y + \partial_x m_{2|1}(z)y(2v+y)]\theta + [\partial_x^2 m_{2|1}(z)v + \partial_x m_{2|1}(z)v(v+2y)]\zeta -$$
$$-2\partial_x\partial_\xi m_{2|1}(z)vy - \partial_\xi m_{2|1}(z)vy(v+y) = 0. \tag{4.14}$$

It follows from Eq. (4.14)

$$\partial_x m_{2|1}(z) = m_{2|1}(z)\overleftarrow{\partial}_\xi = 0 \implies m_{2|1}(z) = \frac{1}{2}m_2 = \text{const.}$$

It is easy to prove that the form $M_{2|2}(f,g)$,

$$M_{2|2}(f,g) = m_2\mu_2(f,g), \quad \mu_2(f,g) = \int dz(-1)^{\varepsilon(f)}\{\partial_x^2 f(z)\}g(z), \quad \epsilon_{\mu_2} = 0. \tag{4.15}$$

satisfies cohomology equation (4.1). Therefore, the form $M_{2|0}(f,g)$ satisfies cohomology equation (4.1) too. That means $M_{2|0}(f,g) = 0$.

Finally, we have proved the following proposition

Proposition 4.7. *Let $n = 1$. Then the general solution of cohomology equation (4.1) in the trivial representation has the form*

$$M_2(f,g) = m_1\mu_1(f,g) + m_2\mu_2(f,g) + d_1^{\text{tr}}M_1(f,g),$$

where

$$\mu_1(f,g) = \int dz(-1)^{\varepsilon(f)}\{\partial_x^3\partial_\xi f(z)\}g(z), \quad \mu_2(f,g) = \int dz(-1)^{\varepsilon(f)}\{\partial_x^2 f(z)\}g(z).$$

The forms $\mu_1(f,g)$ and $\mu_2(f,g)$ are independent nontrivial cocycles. Indeed, suppose that the relation

$$m_1\mu_1(f,g) + m_2\mu_2(f,g) = -d_1^{\text{tr}}M_1(f,g) = \int dz m_1(z)[f(z),g(z)] \tag{4.16}$$

is valid. Take the functions f and g in the form $f(z) \to e^{zp}f(z)$, $g(z) \to e^{-zp}g(z)$:

$$m_1\int dz(-1)^{\varepsilon(f)}\{[(v+\partial_x)^3(\theta+\partial_\xi)f(z)]g(z) + m_2\int dz(-1)^{\varepsilon(f)}\{[(v+\partial_x)^2 f(z)]g(z) =$$

$$= -\int dz m_1(z)\{f(z)(\overleftarrow{\partial}_A + (-1)^{\varepsilon_A}p_A)\omega^{AB}(\partial_B - p_B)g(z)\}.$$

Considering the terms of the fourth order in p, we obtain that Eq. (4.16) can be satisfied for $m_1 = 0$ only. Then, considering the terms of the second order in v, we obtain that Eq. (4.16) can be satisfied for $m_2 = 0$ only.

5 Zeroth and first adjoint cohomologies

5.1 H_{ad}^0

Let $M_0(z)$ be 0-form. Then cohomology equation $0 = d_0^{\mathrm{ad}} M_0(z|f_1) = -[M_0(z), f_1(z)]$ gives $M_0(z) = a = \mathrm{const}$. So $H_{\mathbf{D}_n'}^0 \simeq H_{\mathbf{E}_n}^0 \simeq \mathbb{K}$ and $H_{\mathrm{ad}}^0 = 0$.

5.2 H_{ad}^1

Let $M_1(z|f) = \int du\, m_1(z|u) f(u)$ be 1-form. Then cohomology equation $d_1^{\mathrm{ad}} M_1(z|f,g) = 0$ gives

$$[M_1(z|f), g(z)] - (-1)^{\epsilon(f)\epsilon(g)}[M_1(z|g), f(z)] - M_1(z|[f,g]) = 0 \tag{5.1}$$

Let $z \bigcap \mathrm{supp}(f) = z \bigcap \mathrm{supp}(g) = \varnothing$. Then $\hat{M}_1(z|[f,,g]) = 0$ and

$$\hat{m}_1(z|u)\overleftarrow{\partial}_A \omega_\lambda^{AB}\partial_B f(z) + 2\hat{m}_1(z|u)\Delta f(u) = 0.$$

So $\hat{m}_1(z|u) = 0$, and

$$M_1(z|f) = \sum_{q=0}^{Q} t^{(B)q}(z)(\partial_B)^q f(z), \quad \varepsilon(t^{(B)q}(z)(\partial_B)^q) = \varepsilon_{M_1},$$

Let $f(z) = e^{zp}$, $g(z) = e^{zk}$ in some vicinity of x, and

$$F(z,p) = \sum_{q=0}^{Q} t^{(B)q}(z)(p_B)^q, \quad \epsilon(F(z,p)) = \epsilon_{M_1}.$$

The function $F(z,p)$ is polynomial in p for every z and a degree of this polynomial locally does not depend on z.

Then the cohomology equation acquires the form

$$(F(z,p) + F(z,k) - F(z,p+k))\langle p,k\rangle + [F(z,k), zp] + [F(z,p), zk] = 0. \tag{5.2}$$

Consider the terms of highest order $Q+2$ in Eq. (5.2).
Let $Q \geq 2$. We obtain

$$(F_Q(z,p) + F_Q(z,k) - F_Q(z,p+k))\langle p,k\rangle = 0, \tag{5.3}$$

where $F_Q(z,p) = t^{(B)Q}(z)(p_B)^Q$. Acting on Eq. (5.3) by the operator $\left.\frac{\overleftarrow{\partial}}{\partial k^A}\frac{\overleftarrow{\partial}}{\partial k^B}\right|_{k=0}$, we find

$$F_Q(z,p)\frac{\overleftarrow{\partial}}{\partial p_A}p_C\omega^{CB} + (-1)^{\varepsilon_A\varepsilon_B}F_Q(z,p)\frac{\overleftarrow{\partial}}{\partial p_B}p_C\omega^{CA} = 0. \tag{5.4}$$

The general solution of (5.4) has the form

$$F_Q(z,p)\frac{\overleftarrow{\partial}}{\partial p_A} = t(z,p)(-1)^{\varepsilon_C}p_C\omega^{CA}, \tag{5.5}$$

where $t(z, p)$ is some polynomial in p. Using the property

$$F_Q(z,p)\frac{\overleftarrow{\partial}}{\partial p_A}\frac{\overleftarrow{\partial}}{\partial p_B} - (-1)^{\varepsilon_A \varepsilon_B} F_Q(z,p)\frac{\overleftarrow{\partial}}{\partial p_B}\frac{\overleftarrow{\partial}}{\partial p_A} \equiv 0,$$

we obtain from Eq. (5.5)

$$t(z,p)\frac{\overleftarrow{\partial}}{\partial p_A}p_C\omega^{CB} - (-1)^{\varepsilon_A \varepsilon_B}t(z,p)\frac{\overleftarrow{\partial}}{\partial p_B}p_C\omega^{CA} = 0$$

which implies $t(z,p)\frac{\overleftarrow{\partial}}{\partial p_A}p_A = 0$ and as a consequence $t(z,p) = t(z)$. So, we have

$$Q = 2, \ F_2(z,p) = \frac{1}{2}t(z)\langle p, p\rangle, \ F(z,p) = t^0(z) + t^A(z)p_A + \frac{1}{2}t(z)\langle p, p\rangle,$$

and Eq. (5.2) acquires the form

$$t^0(z)\langle p, k\rangle + [F(z,k), zp] + [F(z,p), zk] = 0. \tag{5.6}$$

Considering the terms of third order in p and k in Eq. (5.6), we obtain $t(z)\frac{\overleftarrow{\partial}}{\partial z^A} = 0$ and so $t(z) = t = \text{const}$. Considering the terms of first order in p and k in Eq. (5.6), we obtain $t^0(z)\frac{\overleftarrow{\partial}}{\partial z^A} = 0$ and so $t^0(z) = t^0 = \text{const}$.

In such a way, Eq. (5.6) is reduced to the equation

$$t^0\omega^{AB} + t^B(z)\frac{\overleftarrow{\partial}}{\partial z^C}\omega^{CA}(-1)^{\varepsilon_A+\varepsilon_B} + t^A(z)\frac{\overleftarrow{\partial}}{\partial z^C}\omega^{CB}(-1)^{\varepsilon_A \varepsilon_B} = 0$$

general solution of which has the form

$$t^A(z) = -\frac{1}{2}t^0 z^A + t_1(z)\frac{\overleftarrow{\partial}}{\partial z^B}\omega^{BA}.$$

Finally, we have obtained that general solution of the cohomology equation (5.1) has the form

$$M_1(z|f) = t^0 \mathcal{E}_z f(z) + t\Delta f(z) + d_0^{\text{ad}} M_0(z|f),$$

where $M_0(z) = t_1(z)$. Each summand in this expression satisfies the cohomology equation and first two of them are nontrivial cocycles. Indeed, it is obvious that an equation

$$t^0 \mathcal{E}_z f(z) + t\Delta f(z) = [\phi(z), f(z)]$$

has solution for $t^0 = t = 0$ only.

Let us discuss the term $[t(z), f(z)]$. Is this expression a coboundary or not? Analogously to [10], the answer depends on the functional class $\mathcal{A}$ in which the considered multilinear forms take their values.

1) $\mathcal{A} = \mathbf{D}'_n$. In this case $t(z) \in \mathbf{D}'_n$ and the form $[t(z), f(z)]$ is exact.

2) $\mathcal{A} = \mathbf{E}_n$. In this case $t(z) \in \mathbf{E}_n$ and the form $[t(z), f(z)]$ is exact.

3) $\mathcal{A} = \mathbf{D}_n$. In this case the condition $[t(z), f(z)] \in \mathbf{D}_n$ gives the restriction $t(z) \in \mathbf{E}_n$ only, and the form $[t(z), f(z)]$ is exact if and only if $t(z) \in \mathbf{D}_n \oplus \mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n)$. So the forms $[t(z), f(z)]$ are independent nontrivial cocycles parametrized by the elements of factor-space $\mathbf{E}_n/\mathbf{Z}_n$, where $\mathbf{Z}_n = \mathbf{D}_n \oplus \mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n)$. Here $\mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n)$ is a centralizer of $\mathbf{D}_n$ in $\mathbf{E}_n$. Evidently, $\mathcal{C}_{\mathbf{E}_n}(\mathbf{D}_n) = \mathbb{K}$.

6 Second adjoint cohomology

For the bilinear form

$$M_2(z|f,g) = \int dv du \, m_2(z|u,v) f(u) g(v) \in \mathcal{A}$$

the cohomology equation has the form

$$d_2^{\mathrm{ad}} M_2(z|f,g,h) = -(-1)^{\epsilon(f)\epsilon(h)}\{(-1)^{\epsilon(f)\epsilon(h)}[M_2(z|f,g),h(z)] +$$
$$+(-1)^{\epsilon(f)\epsilon(h)} M_2(z|[f,g],h) + \mathrm{cycle}(f,g,h)\} = 0. \tag{6.1}$$

6.1 Nonlocal part

6.1.1 $n \geq 2$

Here we prove the following proposition:

Proposition 6.1. *Let $n \geq 2$. Then any adjoint 2-cocycle can be expressed in the form*

$$M_2(z|f,g) = M_{2|\mathrm{loc}}(z|f,g) + d_1^{\mathrm{ad}} M_{1|1}(z|f,g),$$

where $M_{2|\mathrm{loc}} \in \mathcal{M}_0$.

Proof.

Firstly, let us prove that

$$M_2(z|f,g) = M_{2|2}(z|f,g) + d_1^{\mathrm{ad}} M_{1|1}(z|f,g),$$

where $M_{2|2} \in \mathcal{M}_2$. Let

$$z \bigcap \left[\mathrm{supp}(f) \bigcup \mathrm{supp}(g) \bigcup \mathrm{supp}(h)\right] = \mathrm{supp}(f) \bigcap \left[\mathrm{supp}(g) \bigcup \mathrm{supp}(h)\right] = \varnothing.$$

We have $\hat{M}_2(z|f,[g,h]) = 0$, which implies $\hat{m}_2(z|u,v) \overleftarrow{\partial^v}_A \omega^{AB} \partial_B g(v) + 2\hat{m}_2(z|u,v)\Delta g(v) = 0$ and so $\hat{m}_2(z|u,v) = 0$.

As before, we can decompose

$$M_2(z|f,g) = M_{2|1}(z|f,g) + M_{2|2}(z|f,g),$$

where $M_{2|1} \in \mathcal{M}_1$ and $M_{2|2} \in \mathcal{M}_2$.

The proposition 4.1 can be applied to this case also, and we can assume, that the summation in the expression (3.5) for $M_{2|1}(z|f,g)$ is made over even q.

Let

$$z \bigcap \left[\mathrm{supp}(f) \bigcup \mathrm{supp}(g) \bigcup \mathrm{supp}(h)\right] = \varnothing.$$

In this domain, $\hat{M}_{2|2} = 0$ and we obtain

$$\hat{M}_{2|1}(z|[f,g],h) - (-1)^{\epsilon(h)\epsilon(g)} \hat{M}_{2|1}(z|[f,h],g) - \hat{M}_{2|1}(z|f,[g,h]) = 0. \tag{6.2}$$

This equation can be solved analogously to Eq. (4.1).

Using the proposition 4.2 we can write (up to local form which is included to $M_{2|2}(z|f,g)$)

$$M_{2|1}(z|f,g) = \frac{1}{2}\int dum^{1AB}(z|u)(-1)^{\varepsilon(f)}\{[\partial_A^u\partial_B^u f(u)]g(u) + (-1)^{\varepsilon(f)\varepsilon(g)}[\partial_A^u\partial_B^u g(u)]f(u)\} +$$

$$+ \int dum^{10}(z|u)(-1)^{\varepsilon(f)}f(u)g(u).$$

Represent $M_{2|1}(z|f,g)$ in the form

$$M_{2|1}(z|f,g) = M'_{2|1}(z|f,g) + d_1^{\mathrm{ad}}M_{1|1}(z|f,g),\ M_{1|1}(z|f) = \int dum^1(z|u)f(u),$$

$$M'_{2|1}(z|f,g) = \frac{1}{2}\int dum'^{1AB}(z|u)(-1)^{\varepsilon(f)}\{[\partial_A^u\partial_B^u f(u)]g(u) + (-1)^{\varepsilon(f)\varepsilon(g)}[\partial_A^u\partial_B^u g(u)]f(u)\} +$$

$$+ \int dum'^{10}(z|u)(-1)^{\varepsilon(f)}f(u)g(u),$$

$$m'^{1AB}(z|u) = m^{1AB}(z|u) + (-1)^{\varepsilon_A}\omega^{AB}m^1(z|u),\ (-1)^{\varepsilon_A}\omega_{AB}m'^{1AB}(z|u) = 0,$$

$$m'^{10}(z|u) = m^{10}(z|u) + m^1(z|u)\overleftarrow{\Delta},\ \ m^1(z|u) = \frac{1}{2n}(-1)^{\varepsilon_A}\omega_{AB}m^{1AB}(z|u),$$

where ω_{AB} is defined by the relation $\omega_{AB}\omega^{BC} = \delta_A^C$. The form $\hat{M}'_{2|1}(z|f,g)$ satisfies Eq. (6.2),

$$\hat{M}'_{2|1}(z|[f,g],h) - (-1)^{(\varepsilon(h)+1)(\varepsilon(g)+1)}\hat{M}'_{2|1}(z|[f,h],g) - \hat{M}'_{2|1}(z|f,[g,h]) = 0,$$

and, from the proposition 4.3 it follows $\hat{m}'^{1AB}(z|u)p_A p_B = \frac{1}{2}\hat{b}_1(z|u)\langle p,p\rangle$. So $\hat{m}'^{1AB}(z|u) = 0$ and then $\hat{m}'^{10}(z|u) = 0$.

Thus, we have proved that $M_2(z|f,g) = M_{2|2}(z|f,g) + d_1^{\mathrm{ad}}M_{1|1}(z|f,g)$.

Further, let $[z\bigcup \mathrm{supp}(f)\bigcup \mathrm{supp}(g)]\bigcap \mathrm{supp}(h) = \varnothing$. Then the cohomology equation (6.1) gives

$$(-1)^{\epsilon(f)\epsilon_{M_2}}[f(z),\sum_{q=0}^{Q}\int du\hat{m}^{2(A)q}(z|u)(-1)^{\varepsilon(g)}(\partial_A^z)^q g(z)h(u)] -$$

$$-(-1)^{\epsilon(g)\epsilon_{M_2}+\epsilon(f))}[g(z),\sum_{q=0}^{Q}\int du\hat{m}^{2(A)q}(z|u)(-1)^{\varepsilon(f)}(\partial_A^z)^q f(z)h(u)] +$$

$$+\sum_{q=0}^{Q}\int du\hat{m}^{2(A)q}(z|u)(-1)^{\varepsilon(f)+\varepsilon(g)}\{(\partial_A^z)^q[f(z),g(z)]\}h(u) = 0,$$

which implies

$$(-1)^{\epsilon(f)\epsilon(g)+1}\sum_{q=0}^{Q}[\hat{m}^{2(A)q}(z|u)(\partial_A^z)^q g(z),f(z)] +$$

$$+\sum_{q=0}^{Q}[\hat{m}^{2(A)q}(z|u)(\partial_A^z)^q f(z),g(z)] - \sum_{q=0}^{Q}\hat{m}^{2(A)q}(z|u)(\partial_A^z)^q\{[f(z),g(z)]\} = 0. \quad (6.3)$$

Take the functions in the form $f(z) = e^{zp}$, $g(z) = e^{zq}$ in some vicinity of x, and consider the terms of the highest order in p and q which equals to $Q + 2$. We have

$$\hat{m}^{2(A)Q}(z|u)(p_A + k_A)^Q \langle p, k \rangle = 0 \;\Rightarrow\; \hat{m}^{2(A)Q}(z|u) = 0 \;\Rightarrow\; \hat{m}^{2(A)q}(z|u) = 0, \; \forall q.$$

Thus, proposition 6.1 is proved. ∎

6.1.2 $n = 1$

In this case each function $f(z)$ can be decomposed as $f(z) = f_0(x) + \xi f_1(x)$.

Proposition 6.2. *Let $n = 1$. Then any adjoint 2-cocycle can be expressed in the form*

$$M_2(z|f, g) = c_1 m_{2|1}(z|f, g) + c_2 m_{2|2}(z|f, g) + d_1^{\mathrm{ad}} M_1(z|f, g) + M_{2\mathrm{loc}}(z|f, g),$$

where c_i are constants.

The details of the proof can be found in Appendix 1.

6.2 Local part

Consider the local 2-form

$$M_{2|\mathrm{loc}}(z|f, g) = \sum_{a,b=0}^{N} (-1)^{\varepsilon(f)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)[(\partial_A^z)^a f(z)](\partial_B^z)^b g(z),$$

$$m^{(B)_q|(A)_p} = (-1)^{|\varepsilon_A|_{1,p}|\varepsilon_B|_{1,q}} m^{(A)_p|(B)_q}, \;\; \varepsilon(m^{(A)_p|(B)_q}(\partial_A^z)^a(\partial_B^z)^b) = \varepsilon(M_2).$$

The cohomology equation (6.1) reduces to the equation

$$d_2^{\mathrm{ad}} M_{2|\mathrm{loc}}(z|f, g) = 0 =$$

$$= -(-1)^{\epsilon(f)(\epsilon(g)+\epsilon(h))} \Big[\sum_{a,b=0}^{N} (-1)^{\varepsilon(g)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)\{(\partial_A^z)^a g(z)\}(\partial_B^z)^b h(z), f(z)] +$$

$$+(-1)^{\epsilon(g)\epsilon(h)} \Big[\sum_{a,b=0}^{N} (-1)^{\varepsilon(f)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)\{(\partial_A^z)^a f(z)\}(\partial_B^z)^b h(z), g(z)] -$$

$$-[\sum_{a,b=0}^{N} (-1)^{\varepsilon(f)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)\{(\partial_A^z)^a f(z)\}(\partial_B^z)^b g(z), h(z)] -$$

$$- \sum_{a,b=0}^{N} (-1)^{(\varepsilon(f)+\varepsilon(g)+1)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)\{(\partial_A^z)^a [f(z), g(z)]\}(\partial_B^z)^b h(z) +$$

$$+(-1)^{\epsilon(g)(\epsilon(h))} \sum_{a,b=0}^{N} (-1)^{(\varepsilon(f)+\varepsilon(h)+1)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)[(\partial_A^z)^a \{[f(z), h(z)]\}(\partial_B^z)^b g(z) +$$

$$+ \sum_{a,b=0}^{N} (-1)^{\varepsilon(f)(|\varepsilon_B|_{1,b}+1)} m^{(A)_a|(B)_b}(z)\{(\partial_A^z)^a f(z)\}\{(\partial_B^z)^b [g(z), h(z)]\}. \tag{6.4}$$

It is useful to give the expression for the local form, which is a coboundary of a local 1-form $T_{1|\text{loc}}(z|f) = \sum_{a=0}^{K} t^{(A)_p}(z)(\partial_A^z)^a f(z)$, $\varepsilon(t^{(A)_a}(\partial_A^z)^a) = 0$:

$$M_{2|\text{triv}}(z|f,g) = d_1^{\text{ad}} T_{1|\text{loc}}(z|f,g) = [\sum_{a=0}^{K} t^{(A)_p}(z)(\partial_A^z)^a f(z), g(z)] -$$

$$-(-1)^{\epsilon(f)\epsilon(g)}[\sum_{a=0}^{K} t^{(A)_p}(z)(\partial_A^z)^a g(z), f(z)] - \sum_{a=0}^{K} t^{(A)_p}(z)(\partial_A^z)^a [f(z), g(z)]. \qquad (6.5)$$

Let the functions have the form $f(z) = e^{zp}$, $g(z) = e^{zq}$, $h(z) = e^{zr}$ in some vicinity of x. Then Eq. (6.4) acquires the form

$$\Phi(z,p,q,r)\langle p,q\rangle + \Phi(z,q,r,p)\langle q,r\rangle + \Phi(z,r,p,q)\langle r,p\rangle -$$
$$-[F(z,p,q),zr] - [F(z,q,r),zp] - [F(z,r,p),zq] = 0, \qquad (6.6)$$

where

$$\Phi(z,p,q,r) = \Phi(z,q,p,r) = F(z,p+q,r) - F(z,p,r) - F(z,q,r), \qquad (6.7)$$

$$F(z,p,q) = F(z,q,p) = \sum_{a,b=0}^{N} m^{(A)_a|(B)_b}(z)(p_A)^a (q_B)^b.$$

The function $F(z,p,q)$ is polynomial in p, q for every z and a degree of this polynomial locally does not depend on z.

Proposition 6.3. *Up to coboundary*

$$M_{2|\text{loc}}(z|f,g) = c_3 m_{2|3}(z|f,g) + c_4 m_{2|4}(z|f,g) + M'_{2|\text{loc}}(z|f,g),$$

where

$$M'_{2|\text{loc}}(z|e^{zp}, e^{zq})e^{-z(p+q)} \in \mathcal{P}_{1,1}.$$

Proof.

Let $r = 0$. Then Eq. (6.6) reduced to

$$\{F_0(z,p) + F_0(z,q) - F_0(z,p+q)\}\langle p,q\rangle + [F_0(z,p),zq] + [F_0(z,q),zp] = 0,$$

where

$$F_0(z,p) = F(z,p,0) = \sum_{a=0}^{N} m^{(A)_a|0}(z)(p_A)^a.$$

This equation coincides with the equation (5.2) and its solution has the following form

$$m^{0|0}(z) = t^0, \; m^{A|0}(z) = -\frac{1}{2}t^0 z^A + t_1(z)\frac{\overleftarrow{\partial}}{\partial z^B}\omega^{BA}, \; m^{AB|0}(z)\partial_A\partial_B = t\Delta,$$

$$m^{(A)_a|0}(z) = 0, \; a \geq 3, \; t^0 = \text{const}, \; t = \text{const}, \; \epsilon(t_1(z)) = \epsilon_{M_2} + 1.$$

Let us note that

1.

$$(-1)^{\varepsilon(f)}f(z)g(z) - \frac{1}{2}(-1)^{\varepsilon(f)}\{N_z f(z)\}g(z) - \frac{1}{2}(-1)^{\varepsilon(f)}f(z)N_z g(z) =$$

$$= m_{2|3}(z|f,g) + d_1^{\mathrm{ad}}T_{11}(z|f,g) - (-1)^{\varepsilon(f)}\{N_\xi f(z)\}N_\xi g(z) - \frac{\langle z,z\rangle}{4}[f(z),g(z)],$$

where N_z is Euler operator and $T_{11}(z|f) = \frac{\langle z,z\rangle}{4}f(z)$.

2.

$$(-1)^{\varepsilon(f)}\{\Delta f(z)\}g(z) + f(z)\Delta g(z) =$$

$$= m_{2|4}(z|f,g) + \frac{1}{2}(-1)^{\varepsilon(f)}\{\Delta f(z)\}N_z g(z) + \frac{1}{2}\{N_z f(z)\}\Delta g(z),$$

3.

$$(-1)^{\varepsilon(f)}m^{A|(B)_0}(z)\frac{\partial f(z)}{\partial z^A}g(z) + (-1)^{\varepsilon(f)(\varepsilon_A+1)}m^{(B)_0|A}(z)f(z)\frac{\partial g(z)}{\partial z^A} =$$

$$= (-1)^{\varepsilon(f)}[t_1(z),f(z)]g(z) + (-1)^{\varepsilon(f)+\varepsilon(f)\varepsilon(g)}[t_1(z),g(z)]f(z) =$$

$$= d_1^{\mathrm{ad}}T_{12}(z|f,g) - t_1(z)[f(z),g(z)], \quad T_{12}(z|f) = t_1(z)f(z), \quad \varepsilon_{T_1} = \varepsilon(t_1).$$

Choosing $t^{(A)_0}(z) = t_1(z)$ in Exp. (6.5), we obtain (up to coboundary)

$$M_{2|\mathrm{loc}}(z|f,g) = c_3 m_{2|3}(z|f,g) + c_4 m_{2|4}(z|f,g) + M'_{2|\mathrm{loc}}(z|f,g),$$

$$M'_{2|\mathrm{loc}}(z|f,g) = \sum_{a,b=1}^{N}(-1)^{\varepsilon(f)(|\varepsilon_B|_{1,b}+1)}m^{(A)_a|(B)_b}(z)[(\partial_A)^a f(z)](\partial_B)^b g(z), \quad c_3 = t^0, \ c_4 = t.$$

∎

Proposition 6.4. *Up to coboundary*

$$M_{2|\mathrm{loc}}(z|f,g) = c_3 m_{2|3}(z|f,g) + c_4 m_{2|4}(z|f,g) + M''_{2|\mathrm{loc}}(z|f,g),$$

where

$$M''_{2|\mathrm{loc}}(z|e^{zp}, e^{zq})e^{-z(p+q)} \in \mathcal{P}_{2,2}.$$

Proof. Considering in Eq. (6.6) the linear in r terms, we obtain

$$[F^A(z,p+q) - F^A(z,p) - F^A(z,q)]\langle p,q\rangle - \tag{6.8}$$

$$-(-1)^{\varepsilon_A}F(z,p,q)\overleftarrow{\partial^z}_C\omega^{CA} - [F^A(z,q),zp] - [F^A(z,p),zq] = 0, \tag{6.9}$$

$$F^A(z,p) = F(z,p,q)\overleftarrow{\partial}_{q^A}\Big|_{q=0} = \sum_{a=1}^{N}m^{(B)_a|A}(z)(p_B)^a.$$

The linear in q terms in (6.9) give:

$$(-1)^{\varepsilon_A+\varepsilon_B+\varepsilon_A\varepsilon_B}F^B(z,p)\overleftarrow{\partial^z}_C\omega^{CA} + (-1)^{\varepsilon_B+\varepsilon_A\varepsilon_B}m^{A|B}\overleftarrow{\partial^z}_C\omega^{CD}p_D + F^A(z,p)\overleftarrow{\partial^z}_C\omega^{CB} = 0. \tag{6.10}$$

The linear in p terms in (6.10) give:

$$(-1)^{\varepsilon_A+\varepsilon_B+\varepsilon_B\varepsilon_C}m^{A|B}(z)\overleftarrow{\partial}_D\omega^{DC} + (-1)^{\varepsilon_A+\varepsilon_C+\varepsilon_A\varepsilon_B}m^{C|A}(z)\overleftarrow{\partial}_C\omega^{CB} +$$
$$+(-1)^{\varepsilon_B+\varepsilon_C+\varepsilon_A\varepsilon_C}m^{B|C}(z)\overleftarrow{\partial}_C\omega^{CA} = 0. \tag{6.11}$$

Multiply Eq. (6.11) by $(-1)^{\varepsilon_A\varepsilon_C}\varkappa_A\varkappa_B\varkappa_C$, $\varepsilon(\varkappa_A) = \varepsilon_A$, from the right, we obtain

$$m(z,\varkappa)\overleftarrow{d} = 0, \ m(z,\varkappa) = m^{A|B}(z)\varkappa_A\varkappa_B, \ \overleftarrow{d} = \overleftarrow{\partial}_C\omega^{CA}\varkappa_A, \ \overleftarrow{d}\,\overleftarrow{d} = 0,$$

from what it follows in the standard way

$$m(z,\varkappa) = t(z,\varkappa)\overleftarrow{d}, \ t(z,\varkappa) = 2t^A(z)\varkappa_A \implies$$
$$m^{A|B}(z) = (-1)^{\varepsilon_A+\varepsilon_A\varepsilon_B}t^A(z)\overleftarrow{\partial}_C\omega^{CB} + (-1)^{\varepsilon_B}t^B(z)\overleftarrow{\partial}_C\omega^{CA}.$$

The nonlinear in p terms in (6.10) give:

$$(-1)^{\varepsilon_A}F'^A(z,p)\overleftarrow{\partial^z}_C\omega^{CB} + (-1)^{\varepsilon_B+\varepsilon_A\varepsilon_B}F'^B(z,p)\overleftarrow{\partial^z}_C\omega^{CA} = 0, \tag{6.12}$$
$$F'^A(z,p) = \sum_{a=2}^N m^{(B)_a|A}(z)(p_B)^a.$$

Multiplying Eq. (6.12) from the right by $\varkappa_B\varkappa_A$ we obtain $F'(z,p,\varkappa)\overleftarrow{d} = 0$, $F'(z,p,\varkappa) = F'^A(z,p)\varkappa_A$, from what it follows in the standard way

$$F'(z,p,\varkappa) = t'(z,p)\overleftarrow{d}, \ t'(z,p) = \sum_{a=2}^N t^{(A)_p}(z)(p_A)^a \implies$$

$$F'^A(z,p) = t'(z,p)\overleftarrow{\partial^z}_C\omega^{CA}.$$

So, the terms in $M'_{2|\text{loc}}(z|f,g)$ proportional to ∂f, or ∂g, or $\partial f\partial g$ have the following structure

$$\sum_{a=2}^N t^{(A)_a}\overleftarrow{\partial^z}_C\omega^{CB}(-1)^{(\varepsilon_B+1)|\varepsilon_A|_{2,N}+\varepsilon(f)(\varepsilon_B+1)}[(\partial_A)^a f(z)]\partial_B g(z) +$$

$$\sum_{a=2}^N t^{(B)_a}\overleftarrow{\partial^z}_C\omega^{CA}(-1)^{|\varepsilon_B|_{2,N}+\varepsilon(f)(\varepsilon_B|_{2,N}+1)}\partial_A f(z)[(\partial_B)^a g(z)] +$$

$$+(-1)^{\varepsilon(f)(\varepsilon_B+1)}\{(-1)^{\varepsilon_A+\varepsilon_A\varepsilon_B}t^A(z)\overleftarrow{\partial}_C\omega^{CB} + (-1)^{\varepsilon_B}t^B(z)\overleftarrow{\partial}_C\omega^{CA}\}[\partial_A f(z)]\partial_B g(z)$$

$$= d_1^{\text{ad}}T_1(z|f,g) + \text{more}, \ T_1(z|f) = \sum_{a=1}^N t^{(A)_a}(\partial_A)^a f(z),$$

and "more" means the terms proportional to $\partial^a f\partial^b g$ with $a,b \geq 2$. Thus, we obtain (up to coboundary)

$$M_{2|\text{loc}}(z|f,g) = c_3 m_{2|3}(z|f,g) + c_4 m_{2|4}(z|f,g) + M''_{2|\text{loc}}(z|f,g),$$

$$M''_{2|\text{loc}}(z|f,g) = \sum_{a,b=2}^N (-1)^{\varepsilon(f)(|\varepsilon_B|_{1,b}+1)}m^{(A)_a|(B)_b}(z)[(\partial_A)^a f(z)](\partial_B)^b g(z),$$

The cohomology equation takes the form of Eq. (6.6) where now

$$F(z, p, q) = F(z, q, p) \in \mathcal{P}_{2,2}.$$

Proposition 6.5. *If $F(z, p, q)$ is solution of Eq. (6.6) and $F(z, p, q) \in \mathcal{P}_{2,2}$ then $F(z, p, q)$ does not depend on z.*

Proof. Considering in Eq. (6.6) the linear in r terms, we obtain

$$F(z, p, q) \overleftarrow{\partial^z}_C = 0 \implies m^{(A)_a |(B)_b}(z) = m^{(A)_a |(B)_b} = \text{const.}$$

The cohomology equation takes the form

$$\Phi(p, q, r) \langle p, q \rangle + \Phi(q, r, p) \langle q, r \rangle + \Phi(r, p, q) \langle r, p \rangle = 0, \tag{6.13}$$
$$\Phi(p, q, r) = \Phi(q, p, r) = F(p + q, r) - F(p, r) - F(q, r),$$
$$F(p, q) = F(q, p) \in \mathcal{P}_{2,2}.$$

Acting on Eq. (6.13) by the operator $\overleftarrow{\partial}_{r_A} \overleftarrow{\partial}_{r_B} \Big|_{r=0}$ we obtain

$$F(p, q) \left(\overleftarrow{L}_p^{AB} + \overleftarrow{L}_q^{AB} \right) = [F^{AB}(p + q) - F^{AB}(p) - F^{AB}(q)] \langle p, q \rangle, \tag{6.14}$$

where

$$F^{AB}(p) = F(p, q) \overleftarrow{\partial}_{q_A} \overleftarrow{\partial}_{q_B} \Big|_{q=0},$$
$$\overleftarrow{L}_p^{AB} = \overleftarrow{\partial}_{p_A} p_C \omega^{CB} + (-1)^{\varepsilon_A \varepsilon_B} \overleftarrow{\partial}_{p_B} p_C \omega^{CA}, \quad \varepsilon \left(\overleftarrow{L}_p^{AB} \right) = \varepsilon_A + \varepsilon_B + 1. \tag{6.15}$$

It follows from Eq. (6.14) that

$$[(-1)^{\varepsilon_A + \varepsilon_B} F^{AB}(p)] \overleftarrow{L}_p^{CD} - (-1)^{(\varepsilon_C + \varepsilon_D + 1)(\varepsilon_A + \varepsilon_B + 1)} [(-1)^{\varepsilon_C + \varepsilon_D} F^{CD}(p)] \overleftarrow{L}_p^{AB}$$
$$= -\{[(-1)^{\varepsilon_A + \varepsilon_D} F^{AD}(p)] \omega^{BC} + (-1)^{\varepsilon_C \varepsilon_D} [(-1)^{\varepsilon_A + \varepsilon_C} F^{AC}(p)] \omega^{BD} +$$
$$+ (-1)^{\varepsilon_A \varepsilon_B} [(-1)^{\varepsilon_B + \varepsilon_D} F^{BD}(p)] \omega^{AC} + (-1)^{\varepsilon_A \varepsilon_B + \varepsilon_C \varepsilon_D} [(-1)^{\varepsilon_B + \varepsilon_C} F^{BC}(p)] \omega^{AD}\}, \tag{6.16}$$

The operators $\overleftarrow{L}_p^{AB}$ constitute Lie superalgebra $\mathfrak{pe}$.

Proposition 6.6. *The solution of Eq. (6.16) satisfying the condition $F^{AB}(p) \in \mathcal{P}_2$ has the form*

$$F^{AB}(p) = -(-1)^{\varepsilon_A + \varepsilon_B} \varphi(p) \overleftarrow{L}_p^{AB} + c(-1)^{\varepsilon_A} \langle p, p \rangle \omega^{AB},$$

where $\varphi(p)$ is arbitrary polynomial with property $\varphi(p) \in \mathcal{P}_2$ and c is constant.

The proof of this proposition can be found in Appendix 3.

Then, it follows from Eq. (6.14)

$$\{F(p, q) - [\varphi(p + q) - \varphi(p) - \varphi(q)] \langle p, q \rangle\} \left(\overleftarrow{L}_p^{AB} + \overleftarrow{L}_q^{AB} \right) = 0,$$

and one can easily obtain

$$F(p,q) = [\varphi(p+q) - \varphi(p) - \varphi(q) + c_5\langle p,p\rangle + c_5\langle q,q\rangle +$$
$$+ c_6\langle p,p\rangle\langle q,q\rangle]\langle p,q\rangle + c_7\langle p,p\rangle\langle q,q\rangle$$

(c_i are constants), which gives (taking in account the condition $F(p,q) = F(q,p)$)

$$F(p,q) = [\varphi(p+q) - \varphi(p) - \varphi(q) + c_5\langle p,p\rangle + c_5\langle q,q\rangle]\langle p,q\rangle.$$

Then Eq. (6.13) takes the form (all terms including the function φ being cancelled identically)

$$c_5[\langle p,p\rangle\langle q,r\rangle\langle p,q\rangle + \langle q,q\rangle\langle r,p\rangle\langle p,q\rangle + \langle q,q\rangle\langle r,p\rangle\langle q,r\rangle +$$
$$+ \langle r,r\rangle\langle p,q\rangle\langle q,r\rangle + \langle r,r\rangle\langle p,q\rangle\langle r,p\rangle + \langle p,p\rangle\langle q,r\rangle\langle r,p\rangle] = 0,$$

which implies $c_5 = 0$ $\forall n \geq 1$, such that we have

$$F(p,q) = [\varphi(p+q) - \varphi(p) - \varphi(q)]\langle p,q\rangle.$$

Thus, the form $M''_{2|\text{loc}}(z|f,g)$ is equal to

$$M''_{2|\text{loc}}(z|f,g) = [T_1(z|f), g] - (-1)^{(\varepsilon(f)+1)(\varepsilon(g)+1)}[T_1(z|g), f] -$$
$$- [T_1(z|[f,g]) = d_1^{\text{ad}}T_1(z|f,g), \quad T_1(z|f) = \varphi(\partial_z)f(z).$$

Finally, we obtained

$$M_2(z|f,g) = \delta_{1n}[c_1 m_{2|1}(z|f,g) + c_2 m_{2|2}(z|f,g)] +$$
$$+ c_3 m_{2|3}(z|f,g) + c_4 m_{2|4}(z|f,g) + d_1^{\text{ad}}M_1(z|f,g), \tag{6.17}$$

the expressions for the forms $m_{2|1}(z|f,g)$, $m_{2|2}(z|f,g)$, $m_{2|3}(z|f,g)$ and $c_4 m_{2|4}(z|f,g)$ are given by Eqs. (2.4), (2.5), (2.6) and (2.7), respectively.

6.3 Independence and non triviality

All forms $m_{2|a}(z|f,g)$, $a = 1,2,3,4$, are independent nontrivial cocycles for $n = 1$, and forms $m_{2|a}(z|f,g)$, $a = 3,4$, are independent nontrivial cocycles for $n \geq 2$.

6.3.1 $n = 1$

Let

$$c_1 m_{2|1}(z|f,g) + c_2 m_{2|2}(z|f,g) + c_3 m_{2|3}(z|f,g) + c_4 m_{2|5}(z|f,g) = d_1^{\text{ad}}M_1(z|f,g) =$$
$$= [M_1(z|f), g] - (-1)^{\epsilon(f)\epsilon(g)}[M_1(z|g), f] - M_1(z|[f,g]). \tag{6.18}$$

Let

$$[z \cup \text{supp}(f)] \cap \text{supp}(g) = \varnothing \implies [M_1(z|g), f] = 0 \implies \partial_A\hat{M}_1(z|g) = 0 \implies$$

$$M_1(z|f) = M_{1\text{loc}}(z|f) + \int du\theta(x-u)t_{1|1}(u)f_0(u) + \int du\theta(x-u)t_{1|2}(u)f_1(u) + t_{1|3}(f_0) + t_{1|4}(f_1).$$

Let

$$[\text{supp}(f) \cup \text{supp}(g)] > z \implies$$

$$c_1 \int du[\partial_u^3 f_1(u)]g_1(u) = -t_{1|3}([f,g]_0 + [g,f]_0) - t_{1|4}([f,g]_1) \implies$$

$$c_1 = t_{1|3}(f_0) = t_{1|4}(f_1) = 0.$$

Let

$$[\text{supp}(f) \cup \text{supp}(g)] < z \implies$$

$$2c_2 \int du[\partial_u^3 f_1(u)]g_1(u) = \int du t_{1|1}(u)\{[f(u), g(u)]_0 + [g(u), f(u)]_0\} +$$

$$+ \int du\theta(x-u)t_{1|2}(u)[f(u), g(u)]_1 \implies 2c_2 = t_{1|1}(u) = t_{1|2}(u) = 0.$$

Further treatment is common for arbitrary n.

6.3.2 $n \geq 1$, $c_1 = c_2 = 0$

Let

$$c_3 m_{2|3}(z|f,g) + c_4 m_{2|4}(z|f,g) = d_1^{\text{ad}} M_1(z|f,g) =$$
$$= [M_1(z|f), g] - (-1)^{(\varepsilon(f)+1)(\varepsilon(g)+1)}[M_1(z|g), f] - M_1(z|[f,g]). \qquad (6.19)$$

Set

$$z\bigcap\text{supp}(f) = z\bigcap\text{supp}(g) = \varnothing.$$

We have (according to Sec. 5.2)

$$\hat{M}_1(z|[f,,g]) = 0 \implies M_1(z|f) = \sum_{q=0}^{Q} t^{(B)q}(z)(\partial_B)^q f(z).$$

Choosing $g(z) = 1$, we obtain

$$c_3 f(z) - c_3 N_\xi f(z) + c_4 \Delta f(z) = [t^{(B)_0}(z), f(z)] \implies c_3 = c_4 = 0.$$

So, we obtain that Eqs. (6.18) and (6.19) has solutions only if $c_1 = c_2 = c_3 = c_4 = 0$.

6.4 The exactness of the form $M_{d|2}(z|f,g)$

Let us discuss the terms $M_{d|2}(z|f,g)$ in Eq. (6.17),

$$M_{d|2}(z|f,g) \equiv d_1^{\text{ad}} M_1(z|f,g) = [M_1(z|f), g] - (-1)^{(\varepsilon(f)+1)(\varepsilon(g)+1)}[M_1(z|g), f] - M_1(z|[f,g]).$$

Recall once again that the form $M_{d|2}(z|f,g)$ is exact if both functions $M_{d|2}(z|f,g)$ and $M_1(z|g)$ in the expression $d_1^{\text{ad}} M_1(z|f,g)$ belong to the same space $\mathcal{A}$ for all $f, g \in \mathbf{D}_n$.

1) $\mathcal{A} = \mathbf{D}'_n$. In this case $M_1(z|f) \in \mathbf{D}'_n$ and the form $M_{d|2}(z|f, g)$ is exact.

2) $\mathcal{A} = \mathbf{E}_n$. In this case $M_1(z|f) \in \mathbf{E}_n$ and the form $M_{d|2}(z|f, g)$ is exact.[3]

3) $\mathcal{A} = \mathbf{D}_n$. It is easy to prove that if $M_2(z|f, g) \in \mathbf{D}_n$ then $c_1 = c_2 = 0$ and $M_{d|2}(z|f, g) \in \mathbf{D}_n$. Let $f(z) = \omega_{AB} z^B \tilde{f}(z)$, where $\tilde{f}(z) = 1$ for $z \in \mathrm{supp}(g)$. From $M_{d|2}(z|f, g) \in \mathbf{D}_n$, we obtain

$$M_1(z|\partial_A g) \in \mathbf{D}_n, \quad \forall A, g. \tag{6.20}$$

Let $f(z) = [z^A (-1)^{\varepsilon_A} \omega_{AB} z^B / 2n] \tilde{f}(z)$. It follows from $M_{d|2}(z|f, g) \in \mathbf{D}_n$ and (6.20)

$$M_1(z|[f, g]) = M_1(z|\partial_A g'^A) + M_1(z|g) \in \mathbf{D}_n, \quad g'^A = \frac{1}{n} z^A g(z), \quad \Longrightarrow$$

$$M_1(z|g) \in \mathbf{D}_n, \quad \forall g.$$

Thus, the term $M_{d|2}(z|f, g)$ in Eq. (6.17) is the exact form for any space $\mathcal{A}$ unlike the case of the Poisson algebra.

Appendix 1. The proof of Proposition 6.2

Represent the forms $M_1(z|f)$ and $M_2(z|f, g)$ in the form

$$M_1(z|f) = T_{(1)}(x|f_0) + T_{(2)}(x|f_1) + \xi[T_{(3)}(x|f_0) + T_{(4)}(x|f_1)],$$

$$M_2(z|f, g) = M_{(1)}(x|f_0, g_0) + M_{(2)}(x|f_0, g_1) - M_{(2)}(x|g_0, f_1) + M_{(3)}(x|f_1, g_1) +$$
$$+\xi[M_{(4)}(x|f_0, g_0) + M_{(5)}(x|f_0, g_1) - M_{(5)}(x|g_0, f_1) + M_{(6)}(x|f_1, g_1)],$$
$$M_{(1,4)}(x|\varphi, \phi) = M_{(1,4)}(x|\phi, \varphi), \ M_{(3,6)}(x|\varphi, \phi) = -M_{(3,6)}(x|\phi, \varphi).$$

We have for $M_{2d}(z|f, g) \equiv d_1^{\mathrm{ad}} M_1(z|f, g)$:

$$M_{d|(1)}(x|\varphi, \phi) = -T_{(3)}(x|\varphi)\partial_x \phi(x) - T_{(3)}(x|\phi)\partial_x \varphi(x), \ M_{d|(4)}(x|\varphi, \phi) = 0, \quad \text{(A1.1)}$$

$$M_{d|(2)}(x|\varphi, \phi) = \partial_x T_{(1)}(x|\varphi)\phi(x) + T_{(4)}(x|\phi)\partial_x \varphi(x) - T_{(1)}(x|[\varphi, \phi]_0), \quad \text{(A1.2)}$$

$$M_{d|(3)}(x|\varphi, \phi) = \partial_x T_{(2)}(x|\varphi)\phi(x) - \partial_x T_{(2)}(x|\phi)\varphi(x) - T_{(2)}(x|[\varphi, \phi]_1), \quad \text{(A1.3)}$$

$$M_{d|(5)}(x|\varphi, \phi) = \partial_x T_{(3)}(x|\varphi)\phi(x) - T_{(3)}(x|\varphi)\partial_x \phi(x) - T_{(3)}(x|[\varphi, \phi]_0), \quad \text{(A1.4)}$$

$$M_{d|(6)}(x|\varphi, \phi) = \partial_x T_{(4)}(x|\varphi)\phi(x) - \partial_x T_{(4)}(x|\phi)\varphi(x) + T_{(4)}(x|\phi)\partial_x \varphi(x) -$$
$$-T_{(4)}(x|\varphi)\partial_x \phi(x) - T_{(4)}(x|[\varphi, \phi]_1), \quad \text{(A1.5)}$$

$$[\varphi(x), \phi(x)]_0 = \{\partial_x \varphi(x)\}\phi(x), \ [\varphi(x), \phi(x)]_1 = \{\partial_x \varphi(x)\}\phi(x) - \varphi(x)\partial_x \phi(x).$$

It follows from $d_2^{\mathrm{ad}} M_2(z|f_0, g_0, h_0) = 0$ and $d_2^{\mathrm{ad}} M_2(z|f_0, g_0, h_1) = 0$

$$M_{(4)}(x|\varphi, \phi)\partial_x \omega(x) + \mathrm{cycle}(\varphi, \phi, \omega) = 0, \quad \text{(A1.6)}$$
$$M_{(4)}(x|\varphi, \phi)\partial_x \omega(x) - \{\partial_x M_{(4)}(x|\varphi, \phi)\}\omega(x) + M_{(4)}(x|[\varphi, \omega]_0, \phi) + M_{(4)}(x|\varphi, [\phi, \omega]_0) = 0,$$

$$M_{(1)}(x|[\varphi, \omega]_0, \phi) + M_{(1)}(x|\varphi, [\phi, \omega]_0) - \{\partial_x M_{(1)}(x|\varphi, \phi)\}\omega(x) -$$
$$-M_{(5)}(x|\varphi, \omega)\partial_x \phi(x) - M_{(5)}(x|\phi, \omega)\partial_x \varphi(x) = 0. \quad \text{(A1.7)}$$

---

[3] The proof can be found in [10]

It follows from $d_2^{\text{ad}} M_2(z|f_0, g_1, h_1) = 0$

$$M_{(5)}(x|\varphi, \phi)\partial_x\omega(x) - \{\partial_x M_{(5)}(x|\varphi, \phi)\}\omega(x) + \{\partial_x M_{(5)}(x|\varphi, \omega)\}\phi(x) -$$
$$-M_{(5)}(x|\varphi, \omega)\partial_x\phi(x) - M_{(5)}(x|[\varphi, \phi]_0, \omega) + M_{(5)}(x|[\varphi, \omega]_0, \phi) + M_{(5)}(x|\varphi, [\phi, \omega]_1) = 0, \text{ (A1.8)}$$
$$\{\partial_x M_{(2)}(x|\varphi, \omega)\}\phi(x) - \{\partial_x M_{(2)}(x|\varphi, \phi)\}\omega(x) - M_{(2)}(x|[\varphi, \phi]_0, \omega) +$$
$$+M_{(2)}(x|[\varphi, \omega]_0, \phi) + M_{(2)}(x|\varphi, [\phi, \omega]_1) + M_{(6)}(x|\phi, \omega)\partial_x\varphi(x) = 0. \tag{A1.9}$$

It follows from $d_2^{\text{ad}} M_2(z|f_1, g_1, h_1) = 0$

$$M_{(6)}(x|\varphi, \phi)\partial_x\omega(x) - \{\partial_x M_{(6)}(x|\varphi, \phi)\}\omega(x) - M_{(6)}(x|[\varphi, \phi]_1, \omega) +$$
$$+\text{cycle}(\varphi, \phi, \omega) = 0, \tag{A1.10}$$
$$-[\{\partial_x M_{(3)}(x|\varphi, \phi)\}\omega(x) + M_{(3)}(x|[\varphi, \phi]_1, \omega) +$$
$$+\text{cycle}(\varphi, \phi, \omega)] \equiv d_2^{\text{ad}} M_{(3)}(x|\varphi, \phi, \omega) = 0. \tag{A1.11}$$

I. Consider Eq. (A1.6). Let $x \subset U$, U is some fixed bounded domain, $\psi(x) \subset D$ is some fixed function, $\psi(x) = x$ for $x \subset U$. We have from Eq. (A1.6)

$$M_{(4)}(x|\varphi, \phi) = \mu(x|\varphi)\partial_x\phi(x) + \mu(x|\phi)\partial_x\varphi(x), \ \mu(x|\varphi) = -M_{(4)}(x|\varphi, \psi) \implies$$

$$\mu(x|\varphi)\partial_x\phi(x)\partial_x\omega(x) + \mu(x|\phi)\partial_x\varphi(x)\partial_x\omega(x) + \mu(x|\omega)\partial_x\varphi(x)\partial_x\phi(x) = 0 \implies$$

$$\mu(x|\varphi) = \mu(x)\partial_x\varphi(x), \ \mu(x) = -2\mu(x|\psi) \implies$$

$$\mu(x)\partial_x\varphi(x)\partial_x\phi(x)\partial_x\omega(x) = 0 \implies \mu(x) = 0 \implies \mu(x|\varphi) = 0 \implies$$

$$M_{(4)}(x|\varphi, \phi) = 0.$$

II. Consider Eq. (A1.8).
Let $[x \cup \text{supp}(\varphi)] \cap [\text{supp}(\phi) \cup \text{supp}(\omega)] = \varnothing$.
We have $\hat{M}_{(5)}(x|\varphi, [\phi, \omega]_1) = 0 \implies \hat{M}_{(5)}(x|\varphi, \phi) = 0 \implies$

$$M_{(5)}(x|\varphi, \phi) = M_{(5)1}(x|\varphi, \phi) + M_{(5)2}(x|\varphi, \phi),$$
$$M_{(5)1}(x|\varphi, \phi) = \sum_{q=0}^{Q} M_2^q(x|\varphi)\partial_x^q\phi(x), \ M_{(5)2}(x|\varphi, \phi) = \sum_{q=0}^{Q} M_2^q(x|\{\partial^q\varphi\}\phi).$$

Let $[x \cup \text{supp}(\omega)] \cap [\text{supp}(\varphi) \cup \text{supp}(\phi)] = \varnothing$.
We have

$$\sum_{q=0}^{Q} \hat{M}_2^q(x|\{\partial^q\varphi\}\phi)\partial_x\omega(x) - \sum_{q=0}^{Q} \partial_x\hat{M}_2^q(x|\{\partial^q\varphi\}\phi)\omega(x) - \sum_{q=0}^{Q} \hat{M}_1^q(x|[\varphi, \phi]_0)\partial_x^q\omega(x) = 0 \implies$$

$$\hat{M}_1^0(x|[\varphi, \phi]_0) = -\sum_{q=0}^{Q} \partial_x\hat{M}_2^q(x|\{\partial^q\varphi\}\phi), \qquad \hat{M}_1^1(x|[\varphi, \phi]_0) = \sum_{q=0}^{Q} \hat{M}_2^q(x|\{\partial^q\varphi\}\phi);$$

$$\hat{M}_1^q(x|\varphi) = 0, \ \forall q \geq 2 \implies$$

$$M_{(5)2}(x|\varphi, \phi) = -T_{(3)}(x|[\varphi, \phi]_0) + M_{(5)2\text{loc}}(x|\varphi, \phi),$$
$$M_{(5)1}(x|\varphi, \phi) = \partial_x T_{(3)}(x|\varphi)\phi(x) - T_{(3)}(x|\varphi)\partial_x \phi(x) + M_{(5)1\text{loc}}(x|\varphi, \phi),$$
$$\hat{M}_2^q(x|\varphi) = 0, \ q \neq 1, \ T_{(3)}(x|\varphi) = -M_2^1(x|\varphi).$$

So, we obtain

$$M_{(5)}(x|\varphi, \phi) = M_{(5)\text{loc}}(x|\varphi, \phi) + M_{d|(5)}(x|\varphi, \phi),$$

where the expression for $M_{d|(5)}(x|\varphi, \phi)$ is given by Eq. (A1.4).

III. Consider Eq. (A1.7).

Let $[x \cup \text{supp}(\varphi)] \cap [\text{supp}(\phi) \cup \text{supp}(\omega)] = \varnothing$.

We have for $M'_{(1)}(x|\varphi, \phi) = M_{(1)}(x|\varphi, \phi) - M_{d|(1)}(x|\varphi, \phi)$, the expression for $M_{d|(5)}(x|\varphi, \phi)$ is given by Eq. (A1.1),

$$\hat{M}'_{(1)}(x|\varphi, [\phi, \omega]_0) = 0 \implies$$

$$M'_{(1)}(x|\varphi, \phi) = \sum_{q=0}^{Q} M_3^q(x|\varphi)\partial_x^q \phi(x) = \sum_{q=0}^{Q} M_3^q(x|\phi)\partial_x^q \varphi(x) \implies$$

$$M'_{(1)}(x|\varphi, \phi) = M_{(1)\text{loc}}(x|\varphi, \phi), \ M_{(1)}(x|\varphi, \phi) = M_{(1)\text{loc}}(x|\varphi, \phi) + M_{d|(1)}(x|\varphi, \phi).$$

IV. Consider Eq. (A1.10).

Let $[x \cup \text{supp}(\omega)] \cap [\text{supp}(\varphi) \cup \text{supp}(\phi)] = \text{supp}(\varphi) \cap \text{supp}(\phi) = \varnothing$.

We have $\hat{M}_{(6)}(x|\varphi, \phi)\partial_x \omega(x) - \{\partial_x \hat{M}_{(6)}(x|\varphi, \phi)\}\omega(x) = 0$ and so $\hat{M}_{(6)}(x|\varphi, \phi) = 0$. Thus

$$M_{(6)}(x|\varphi, \phi) = M_{(6)4}(x|\varphi, \phi) + M_{(6)5}(x|\varphi, \phi),$$

$$M_{(6)4}(x|\varphi, \phi) = \sum_{q=0}^{Q} \{\partial_x^q \varphi(x) M_4^q(x|\phi) - M_4^q(x|\varphi)\partial_x^q \phi(x)\},$$

$$M_{(6)5}(x|\varphi, \phi) = \sum_{k=0}^{K} M_5^{2k+1}(x|\{\partial^{2k+1}\varphi\}\phi - \varphi\partial^{2k+1}\phi).$$

Let $[x \cup \text{supp}(\omega)] \cap [\text{supp}(\varphi) \cup \text{supp}(\phi)] = \varnothing$.

We have

$$\sum_{k=0}^{K} \hat{M}_5^{2k+1}(x|\{\partial^{2k+1}\varphi\}\phi - \varphi\partial^{2k+1}\phi)\partial_x \omega(x) - \sum_{k=0}^{K} \partial_x \hat{M}_5^{2k+1}(x|\{\partial^{2k+1}\varphi\}\phi - \varphi\partial^{2k+1}\phi)\omega(x) -$$

$$+ \sum_{q=0}^{Q} \hat{M}_4^q(x|\{\partial\varphi\}\phi - \varphi\partial\phi)\partial_x^q \omega(x) = 0.$$

So

$$\hat{M}_4^0(x|\{\partial\varphi\}\phi - \varphi\partial\phi) = \sum_{k=0}^{K} \partial_x \hat{M}_5^{2k+1}(x|\{\partial^{2k+1}\varphi\}\phi - \varphi\partial^{2k+1}\phi),$$

$$\hat{M}_4^1(x|\{\partial\varphi\}\phi - \varphi\partial\phi) = -\sum_{k=0}^{K} \hat{M}_5^{2k+1}(x|\{\partial^{2k+1}\varphi\}\phi - \varphi\partial^{2k+1}\phi),$$

$$\hat{M}_4^q(x|\varphi) = 0 \ \forall q \geq 2. \implies$$

$$M_{(6)4}(x|\varphi, \phi) = \partial_x T_{(4)}(x|\phi)\varphi(x) - \partial_x T_{(4)}(x|\varphi)\phi(x) +$$
$$+ T_{(4)}(x|\phi)\partial_x \varphi(x) - T_{(4)}(x|\varphi)\partial_x \phi(x) + M_{(6)4\text{lok}}(x|\varphi, \phi),$$
$$M_{(6)5}(x|\varphi, \phi) = -T_{(4)}(x|[\varphi, \phi]_1) + M_{(6)5\text{lok}}(x|\varphi, \phi),$$
$$\hat{M}_5^{2k+1}(x|\varphi) = 0, \ k \geq 1, \ T_{(4)}(x|\varphi) = -M_5^1(x|\varphi).$$

So, we obtain

$$M_{(6)}(x|\varphi, \phi) = M_{(6)\text{loc}}(x|\varphi, \phi) + M_{d|(6)}(x|\varphi, \phi),$$

where the expression for $M_{d|(6)}(x|\varphi, \phi)$ is given by Eq. (A1.5).

V. Consider Eq. (A1.9).

We have for $M'_{(2)}(x|\varphi, \phi) = M_{(2)}(x|\varphi, \phi) - T_{(4)}(x|\phi)\partial_x \varphi(x)$

$$\{\partial_x M'_{(2)}(x|\varphi, \omega)\}\phi(x) - \{\partial_x M'_{(2)}(x|\varphi, \phi)\}\omega(x) - M'_{(2)}(x|[\varphi, \phi]_0, \omega) +$$
$$+ M'_{(2)}(x|[\varphi, \omega]_0, \phi) + M'_{(2)}(x|\varphi, [\phi, \omega]_1) + M_{(6)\text{loc}}(x|\phi, \omega)\partial_x \varphi(x) = 0.$$

Let $[x \cup \text{supp}(\varphi)] \cap [\text{supp}(\phi) \cup \text{supp}(\omega)] = \varnothing$.

We have $\hat{M}'_{(2)}(x|\varphi, [\phi, \omega]_1) = 0$. So $\hat{M}'_{(2)}(x|\varphi, \phi) = 0$ and

$$M'_{(2)}(x|\varphi, \phi) = M_{(2)6}(x|\varphi, \phi) + M_{(2)7}(x|\varphi, \phi),$$
$$M_{(2)6}(x|\varphi, \phi) = \sum_{q=0}^{Q} M_6^q(x|\varphi)\partial_x^q \phi(x), \ M_{(2)7}(x|\varphi, \phi) = \sum_{q=0}^{Q} M_7^q(x|\{\partial^q \varphi\}\phi).$$

Let $[x \cup \text{supp}(\omega)] \cap [\text{supp}(\varphi) \cup \text{supp}(\phi)] = \varnothing$.

We obtain $\{\partial_x \hat{M}'_{(2)}(x|\varphi, \phi)\}\omega(x) + \hat{M}'_{(2)}(x|[\varphi, \phi]_0, \omega) = 0$ and so

$$\sum_{q=0}^{Q} \{\partial_x \hat{M}_7^q(x|\{\partial^q \varphi\}\phi)\}\omega(x) + \sum_{q=0}^{Q} \hat{M}_6^q(x|\{\partial\varphi\}\phi)\partial_x^q \omega(x) = 0 \implies$$

$$\sum_{q=0}^{Q} \partial_x \hat{M}_7^q(x|\{\partial^q \varphi\}\phi) + \hat{M}_6^0(x|\{\partial\varphi\}\phi) = 0, \ \hat{M}_6^q(x|\varphi) = 0, \ q \geq 1 \implies$$

$$\hat{M}_6^0(x|\varphi) = -\partial_x \hat{M}_7^1(x|\varphi), \ \partial_x \hat{M}_7^q(x|\varphi) = 0, \ q \neq 1$$

So, we found

$$M_{(2)}(x|\varphi, \phi) = M_{(2)\text{loc}}(x|\varphi, \phi) + M_{d|(2)}(x|\varphi, \phi) + M''_{(2)}(x|\varphi, \phi),$$
$$M''_{(2)}(x|\varphi, \phi) = \sum_{q=0, q\neq 1}^{Q} M_7^q(x|\{\partial^q \varphi\}\phi), \ \partial_x \hat{M}_7^q(x|\varphi) = 0,$$

where the expression for $M_{d|(2)}(x|\varphi, \phi)$ is given by Eq. (A1.2) with $T_{(1)}(x|\phi) = -M_7^1(x|\varphi)$. For $M''_{(2)}(x|\varphi, \phi)$ we obtain

$$\{\partial_x M''_{(2)}(x|\varphi, \omega)\}\phi(x) - \{\partial_x M''_{(2)}(x|\varphi, \phi)\}\omega(x) - M''_{(2)}(x|[\varphi, \phi]_0, \omega) +$$
$$+ M''_{(2)}(x|[\varphi, \omega]_0, \phi) + M''_{(2)}(x|\varphi, [\phi, \omega]_1) = M_{\text{loc}}(x|\varphi, \phi, \omega),$$

where $M_{\text{loc}}(x|\varphi, \phi, \omega)$ is some local over all arguments functional.

Let $x \cap [\text{supp}(\varphi) \cup \text{supp}(\phi) \cup \text{supp}(\omega)] = \varnothing$.

We obtain

$$\hat{M}''_{(2)}(x|[\varphi, \phi]_0, \omega) - \hat{M}''_{(2)}(x|[\varphi, \omega]_0, \phi) - \hat{M}''_{(2)}(x|\varphi, [\phi, \omega]_1) = 0$$

or

$$\sum_{q=0, q\neq 1}^{Q} \hat{M}_7^q(x|\{\partial^q(\partial\varphi\phi)\}\omega - \{\partial^q(\partial\varphi\omega)\}\phi - \{\partial^q\varphi\}[\partial\phi\omega - \phi\partial\omega]) = 0. \qquad \text{(A1.12)}$$

Let $\varphi(x) \to e^{px}\varphi(x)$, $\phi(x) \to e^{kx}\phi(x)$, $\omega(x) \to e^{-(p+k)x}\omega(x)$. Consider the terms of highest order in p, k in Eq. (A1.12),

$$[p(p+k)^Q - p(-k)^Q - (p+2k)p^Q]\hat{M}_7^Q(x|\varphi\phi\omega) = 0 \implies$$

$$\hat{M}_7^Q(x|\varphi) = 0, \ Q \neq 1.$$

Finally, we have

$$M_{(2)}(x|\varphi, \phi) = M_{(2)\text{loc}}(x|\varphi, \phi) + M_{d|(2)}(x|\varphi, \phi).$$

VI. Consider Eq. (A1.11).

Let $[x \cup \text{supp}(\omega)] \cap [\text{supp}(\varphi) \cup \text{supp}(\phi)] = x \cap \text{supp}(\omega) = \varnothing$.

We have

$$\hat{M}_{(3)}(x|[\varphi, \phi]_1, \omega) = 0 \implies \hat{M}_{(3)}(x|\varphi, \omega) = 0 \implies$$

$$\begin{aligned}
M_{(3)}(x|\varphi, \phi) &= M_{(3)8}(x|\varphi, \phi) + M_{(3)9}(x|\varphi, \phi), \\
M_{(3)8}(x|\varphi, \phi) &= \sum_{q=0}^{Q} \{\partial_x^q \varphi(x) M_8^q(x|\phi) - M_8^q(x|\varphi)\partial_x^q \phi(x)\}, \\
M_{(3)9}(x|\varphi, \phi) &= \sum_{l=0}^{L} M_9^{2l+1}(x|\{\partial^{2l+1}\varphi\}\phi - \varphi\partial^{2l+1}\phi).
\end{aligned}$$

Let $[x \cup \text{supp}(\omega)] \cap [\text{supp}(\varphi) \cup \text{supp}(\phi)] = \varnothing$.

We have $\{\partial_x \hat{M}_{(3)}(x|\varphi, \phi)\}\omega(x) + \hat{M}_{(3)}(x|[\varphi, \phi]_1, \omega) = 0$ or $\sum_{l=0}^{L}[\partial_x \hat{M}_9^{2l+1}(x|\{\partial^{2l+1}\varphi\}\phi - \varphi\partial^{2l+1}\phi)]\omega(x) - \sum_{q=0}^{Q} \hat{M}_8^q(x|\partial\varphi\phi - \varphi\partial\phi)\partial_x^q\omega(x)\} = 0$. So

$$\sum_{l=0}^{L}[\partial_x \hat{M}_9^{2l+1}(x|\{\partial^{2l+1}\varphi\}\phi - \varphi\partial^{2l+1}\phi)] - \hat{M}_8^0(x|\partial\varphi\phi - \varphi\partial\phi) = 0,$$

$$\hat{M}_8^q(x|\varphi) = 0, \ q \geq 1 \implies$$

$$\hat{M}_8^0(x|\varphi) = \partial_x \hat{M}_9^1(x|\varphi), \ \partial_x \hat{M}_9^{2l+1}(x|\varphi) = 0, \ l \geq 1.$$

Thus, we found

$$\begin{aligned}
M_{(3)}(x|\varphi, \phi) &= M_{d|(3)}(x|\varphi, \phi) + M'_{(3)}(x|\varphi, \phi) + M_{(3)\text{lok}}(x|\varphi, \phi), \\
M'_{(3)}(x|\varphi, \phi) &= \sum_{l=1}^{L} M_9^{2l+1}(x|\{\partial^{2l+1}\varphi\}\phi - \varphi\partial^{2l+1}\phi),
\end{aligned}$$

where the expression for $M_{d|(3)}(x|\varphi, \phi)$ is given by Eq. (A1.3) with $T_{(2)}(x|\phi) = -M_9^1(x|\varphi)$. For $M_{(3)}'(x|\varphi, \phi)$ we obtain

$$\{\partial_x M_{(3)}'(x|\varphi, \omega)\}\phi(x) - \{\partial_x M_{(3)}'(x|\varphi, \phi)\}\omega(x) - \{\partial_x M_{(3)}'(x|\phi, \omega)\}\varphi(x) -$$
$$- M_{(3)}'(x|[\varphi, \phi]_1, \omega) + M_{(3)}'(x|[\varphi, \omega]_1, \phi) + M_{(3)}'(x|\varphi, [\phi, \omega]_1) = M_{loc}'(x|\varphi, \phi, \omega).$$

where $M_{loc}'(x|\varphi, \phi, \omega)$ is some local over all arguments functional.

Let $x \cap [\mathrm{supp}(\varphi) \cup \mathrm{supp}(\phi) \cup \mathrm{supp}(\omega)] = \varnothing$.

We obtain $\hat{M}_{(3)}'(x|[\varphi, \phi]_1, \omega) + \hat{M}_{(3)}'(x|[\omega, \varphi]_1, \phi) + \hat{M}_{(3)}'(x|[\phi, \omega]_1, \varphi) = 0$ or

$$\sum_{l=1}^{L} \hat{M}_9^{2l+1}(x|\{\partial^{2l+1}[\varphi, \phi]_1\}\omega - \{\partial^{2l+1}\omega\}[\varphi, \phi]_1 + \mathrm{cycle}(\varphi, \phi, \omega)) = 0. \tag{A1.13}$$

Let $\varphi(x) \to e^{px}\varphi(x)$, $\phi(x) \to e^{kx}\phi(x)$, $\omega(x) \to e^{-(p+k)x}\omega(x)$. Consider the terms of highest order in p, k in Eq. (A1.13),

$$[(p-k)(p+k)^{2L+1} + (2p+k)k^{2L+1} - (p+2k)p^{2L+1}]\hat{M}_9^{2L+1}(x|\varphi\phi\omega) = 0 \implies$$

$$\hat{M}_9^{2L+1}(x|\varphi) = 0, \; L \geq 2 \implies$$

$$M_{(3)}'(x|\varphi, \phi) = M_9^3(x|\{\partial^3\varphi\}\phi - \varphi\partial^3\phi),$$
$$\partial_x \hat{M}_9^3(x|\varphi) = 0, \; \partial_x \hat{m}_9^3(x|u) = 0, \tag{A1.14}$$
$$\hat{M}_9^3(x|\{\partial^3[\varphi, \phi]_1\}\omega - \{\partial^3\omega\}[\varphi, \phi]_1 + \mathrm{cycle}(\varphi, \phi, \omega)) = 0, \tag{A1.15}$$
$$M_9^3(x|\varphi) = \int du \, m_9^3(x|u)\varphi(u).$$

Let $\omega(x) = 1$ in Eq. (A1.15),

$$\hat{M}_9^3(x|[2\overleftarrow{\partial}\partial^3\varphi + 3\overleftarrow{\partial}^2\partial^2\varphi + \overleftarrow{\partial}^3\partial\varphi]\phi) = 0 \implies \hat{m}_9^3(x|u)\overleftarrow{\partial_u} = 0. \tag{A1.16}$$

It follows from Eq. (A1.14)

$$\partial_x m_9^3(x|u) = \partial_x \sum_{q=0} m_9^{3q}(x)\partial_x^q \delta(x-u) + \delta(x-u)\mu_{91}^3(u) =$$

$$= \partial_x \left(\sum_{q=0} m_9^{3q}(x)\partial_x^q \delta(x-u) + \theta(x-u)\mu_{91}^3(u) \right) \implies$$

$$m_9^3(x|u) = \sum_{q=0} m_9^{3q}(x)\partial_x^q \delta(x-u) + \theta(x-u)\mu_{91}^3(u) + \mu_{92}^3(u).$$

It follows from Eq. (A1.16)

$$\theta(x-u)\partial_u \mu_{91}^3(u) + \partial_u \mu_{92}^3(u) = 0, \; x \neq u \implies$$

$$\mu_{91}^3(u) = c_2 = \mathrm{const}, \; \mu_{92}^3(u) = \frac{1}{2}c_1 = \mathrm{const}.$$

So, we obtain

$$
\begin{aligned}
M_{(3)}(x|\varphi,\phi) &= c_1 M_{(3)1}(x|\varphi,\phi) + M_{d|(3)}(x|\varphi,\phi) + M''_{(3)}(x|\varphi,\phi) + M_{(3)\mathrm{lok}}(x|\varphi,\phi), \\
M_{(3)1}(x|\varphi,\phi) &= \int du[\partial_u^3 \varphi(u)]\phi(u), \\
M''_{(3)}(x|\varphi,\phi) &= c_2 \int du\,\theta(x-u)[\{\partial_u^3\varphi(u)\}\phi(u) - \varphi(u)\partial_u^3\phi(u)].
\end{aligned}
$$

$$
d_2^{\mathrm{ad}} M''_{(3)}(x|\varphi,\phi,\omega) = -\mu_{91}^3[\{\partial_x^2\varphi(x)\}\phi(x)\partial_x\omega(x) - \{\partial_x^2\varphi(x)\}\{\partial_x\phi(x)\}\omega(x)] +
$$
$$
+\mathrm{cycle}(\varphi,\phi,\omega).
$$

Note that the form

$$
m_{2|1}(x|f,g) = M_{(3)1}(x|f_1,g_1) = \int du[\partial^3 f_1(u)]g_1(u), \ \varepsilon_{m_{2|1}} = 0, \tag{A1.17}
$$

satisfies the cohomology equation (6.1). Furthermore, a form $c_2 m_{2|2}(x|f,g)$, which differs from $M''_{(3)}(x|f_1,g_1)$ by local form, satisfies the cohomology equation (6.1) too,

$$
m_{2|2}(x|f,g) = \int du\,\theta(x-u)[\{\partial_u^3 f_1(u)\}g_1(u) - f_1(u)\partial_u^3 g_1(u)] +
$$
$$
+x[\{\partial_x^2 f_1(x)\}\partial_x g_1(x) - \{\partial_x f_1(x)\}\partial_x^2 g_1(x)], \ \varepsilon_{m_{2|2}} = 0. \tag{A1.18}
$$

So, we obtained

$$
M_2(z|f,g) = c_1 m_{2|1}(x|f,g) + c_2 m_{2|2}(x|f,g) + d_1^{\mathrm{ad}} M_1(z|f,g) + M_{2\mathrm{loc}}(z|f,g).
$$

Appendix 2. Lie Superalgebra PE

Let $p_i = x_i,\ p_{n+\alpha} = \xi_\alpha,\ q_i = y_i,\ q_{n+\alpha} = \eta_\alpha,\ i,j,...,\alpha,\beta,... = 1,2,...,n.$
Let $\overleftarrow{L}^{AB} = \overleftarrow{\partial}_A p_C \omega^{CB} + (-1)^{\varepsilon_A \varepsilon_B}\overleftarrow{\partial}_B p_C \omega^{CA},\ \varepsilon\left(\overleftarrow{L}^{AB}\right) = \varepsilon_A + \varepsilon_B + 1.$ Introduce as well the notation

$$
\begin{aligned}
\overleftarrow{M}_{ij} &= -\overleftarrow{L}^{ij} = \frac{\overleftarrow{\partial}}{\partial x_i}\xi_j + \frac{\overleftarrow{\partial}}{\partial x_j}\xi_i, \\
\overleftarrow{P}_{i\alpha} &= \overleftarrow{L}^{i,n+\alpha} = \frac{\overleftarrow{\partial}}{\partial x_i}x_\alpha - \frac{\overleftarrow{\partial}}{\partial \xi_\alpha}\xi_i, \\
\overleftarrow{Q}_{\alpha\beta} &= \overleftarrow{L}^{n+\alpha,n+\beta} = \frac{\overleftarrow{\partial}}{\partial \xi_\alpha}x_\beta - \frac{\overleftarrow{\partial}}{\partial \xi_\beta}x_\alpha,
\end{aligned}
$$

$$
\overleftarrow{\Sigma}_p = \overleftarrow{P}_{pii} = \overleftarrow{N}_x - \overleftarrow{N}_\xi, \ \overleftarrow{N}_x = \frac{\overleftarrow{\partial}}{\partial x_i}x_i, \ \overleftarrow{N}_\xi = \frac{\overleftarrow{\partial}}{\partial \xi_i}\xi_i.
$$

These operators form the periplectic superalgebra $\mathfrak{pe}(n)$ [11], [7]:

$$
\left[\overleftarrow{L}^{AB}, \overleftarrow{L}^{CD}\right] =
$$
$$
-\left(\omega^{BC}\overleftarrow{L}^{AD} + (-1)^{\varepsilon_C \varepsilon_D}\omega^{BD}\overleftarrow{L}^{AC} + (-1)^{\varepsilon_A \varepsilon_B}\omega^{AC}\overleftarrow{L}^{BD} + (-1)^{\varepsilon_A \varepsilon_B + \varepsilon_C \varepsilon_D}\omega^{AD}\overleftarrow{L}^{BC}\right),
$$

and $\overleftarrow{\Sigma}_p$ generates $U(1)$ center of the even subalgebra of this algebra. Algebra $\mathfrak{spe}(n) = \mathfrak{pe}(n)/U(1)$ is also known as strange Lie superalgebra $P(n-1)$ [12].

We have $z_A(\overleftarrow{L}_p^{BC} + \overleftarrow{L}_q^{BC}) = z_A\overleftarrow{L}_z^{BC}$, $z_A = p_A + q_A$, $\langle p, p \rangle \overleftarrow{L}_p^{AB} = \langle p, q \rangle (\overleftarrow{L}_p^{BC} + \overleftarrow{L}_q^{BC}) = 0$, where

$$\langle p, q \rangle = p_A(-1)^{\varepsilon_A}\omega^{AB}q_B = \langle q, p \rangle .$$

Appendix 3. The proof of Proposition 6.6

Here and in successive appendices we solve the equation

$$F^{AB}(p)\overleftarrow{L}_p^{CD} - (-1)^{(\varepsilon_C+\varepsilon_D+1)(\varepsilon_A+\varepsilon_B+1)}F^{CD}(p)\overleftarrow{L}_p^{AB} = -\{F^{AD}(p)\omega^{BC} +$$
$$+(-1)^{\varepsilon_C\varepsilon_D}F^{AC}(p)\omega^{BD} + (-1)^{\varepsilon_A\varepsilon_B}F^{BD}(p)\omega^{AC} + (-1)^{\varepsilon_A\varepsilon_B+\varepsilon_C\varepsilon_D}F^{BC}(p)\omega^{AD}\}\text{(A3.1)}$$

or

$$U^{ij}\overleftarrow{M}_{kl} + U^{kl}\overleftarrow{M}_{ij} = 0, \tag{A3.2}$$

$$U^{ij}\overleftarrow{P}_{k\alpha} + U^{ik}\delta_{j\alpha} + U^{jk}\delta_{i\alpha} = -V^{k\alpha}\overleftarrow{M}_{ij}, \tag{A3.3}$$

$$U^{ij}\overleftarrow{Q}_{\alpha\beta} - W^{\alpha\beta}\overleftarrow{M}_{ij} = V^{i\alpha}\delta_{j\beta} + V^{j\alpha}\delta_{i\beta} - V^{i\beta}\delta_{j\alpha} - V^{j\beta}\delta_{i\alpha}, \tag{A3.4}$$

$$V^{i\alpha}\overleftarrow{P}_{j\beta} - V^{j\beta}\overleftarrow{P}_{i\alpha} + V^{j\alpha}\delta_{i\beta} - V^{i\beta}\delta_{j\alpha} = 0, \tag{A3.5}$$

$$W^{\alpha\beta}\overleftarrow{P}_{i\gamma} - W^{\alpha\gamma}\delta_{i\beta} + W^{\beta\gamma}\delta_{i\alpha} = V^{i\gamma}\overleftarrow{Q}_{\alpha\beta}, \tag{A3.6}$$

$$W^{\alpha\beta}\overleftarrow{Q}_{\gamma\delta} + W^{\gamma\delta}\overleftarrow{Q}_{\alpha\beta} = 0, \tag{A3.7}$$

where we used a notation

$$U^{ij} \equiv F^{ij},\ V^{i\alpha} \equiv F^{i,n+\alpha},\ W^{\alpha\beta} \equiv F^{n+\alpha,n+\beta}.$$

I. It is obviously that Eq. (A3.1) has solutions of the form

$$F^{AB}(p) = F_1^{AB}(p) = f(p)\overleftarrow{L}_p^{AB} \tag{A3.8}$$

where f is an arbitrary polynomial in p.

II. The second solution has the form

$$\begin{aligned} F^{AB}(p) = F_2^{AB}(p) &= (v_1 + v_2\langle p, p \rangle)(-1)^{\varepsilon_A}\omega^{AB} = \\ &= (U_2^{ij} = 0,\ V_2^{i\alpha}(p) = (v_1 + v_2\langle p, p \rangle)\delta_{i\alpha},\ W_2^{\alpha\beta} = 0), \quad \text{(A3.9)} \end{aligned}$$

where v_i are constant. Note that $F_2^{AB}(p)$ can not be represented in the form (A3.8).

Lemma A3.1.

General solution of Eq. (A3.1) is

$$F^{AB}(p) = F_1^{AB}(p) + F_2^{AB}(p), \tag{A3.10}$$

where $F_1(p)$ and $F_2(p)$ are defined by Eqs. (A3.8) and (A3.9).

Appendix 4. The proof of Proposition A3.1 for $n = 1$.

In this Appendix we omit everywhere the only possible index 1. Because $n = 1$, Eqs. (A3.2) - (A3.7) reduces to

$$W = 0$$
$$\xi \partial_x U = 0$$
$$x\partial_x U - \xi\partial_\xi U + 2U = -2\xi\partial_x V. \tag{A4.1}$$

So $U = u_1(x)\xi + u_0$. Let $V = \xi v_1(x) + v_0(x)$. Then Eq. (A4.1) gives $u_0 = 0$ and $x\partial_x u_1 + u_1 = -2\partial_x v_0$, which implies $v_0(x) = -\frac{1}{2}xu_1 + c$.

Let

$$f = -\frac{1}{2}\int_0^x u_1\,dx + \xi(x\partial_x - 1)^{-1}[v_1(x) - x\partial_y v_1(y)|_{y=0}].$$

Then $U = -2\xi\partial_x f$, $V = (x\partial_x - \xi\partial_\xi)f + c + 2c_1\xi x$, where $c_1 = -\frac{1}{2}\partial_y v_1(y)|_{y=0}$.

Appendix 5. The proof of Proposition A3.1 for $n = 2$.

Equation (A3.7): As W is antisymmetric, the only nonzero element is $W^{12} = -W^{21} \equiv w$. It follows from (A3.7) that $wQ^{12} = 0$ Let $w = w_{12}\xi_1\xi_2 + w_1\xi_1 + w_2\xi_2 + w_0$. Then $w\left(\overleftarrow{\frac{\partial}{\partial\xi_1}}x_2 - \overleftarrow{\frac{\partial}{\partial\xi_2}}x_1\right) = -w_{12}x_2\xi_2 - w_{12}x_1\xi_1 + w_1x_2 - w_2x_1 = 0$. So $w_{12} = 0$ and $w_1x_2 - w_2x_1 = 0$, which implies $w_i = x_iW(x)$. Thus $W^{12} = W(x)(x_1\xi_1 + x_2\xi_2) + w_0(x) = (W(x)\xi_2\xi_1)Q^{12} + w_0(x)$.

Up to (A3.8) we have

$$W^{12}(x, \xi) = w(x)$$

Equation (A3.6): As $W^{\alpha\beta}$ does not depend on ξ, we have $x_\gamma\frac{\partial}{\partial x_i}W^{\alpha\beta} - W^{\alpha\gamma}\delta_{i\beta} + W^{\beta\gamma}\delta_{i\alpha} = V^{i\gamma}\overleftarrow{Q}_{\alpha\beta}$. Consider all 4 cases

$\alpha = 1$, $\beta = 2$, $i = 1$, $\gamma = 1$:

$$x_1\frac{\partial}{\partial x_1}w - w = V^{11}\overleftarrow{Q}^{12} \tag{A5.1}$$

$\alpha = 1$, $\beta = 2$, $i = 2$, $\gamma = 2$:

$$x_2\frac{\partial}{\partial x_2}w - w = V^{22}\overleftarrow{Q}^{12} \tag{A5.2}$$

$\alpha = 1$, $\beta = 2$, $i = 2$, $\gamma = 1$:

$$x_2\frac{\partial}{\partial x_1}w = V^{12}\overleftarrow{Q}^{12} \tag{A5.3}$$

$\alpha = 1$, $\beta = 2$, $i = 1$, $\gamma = 2$:

$$x_1\frac{\partial}{\partial x_2}w = V^{21}\overleftarrow{Q}^{12} \tag{A5.4}$$

All other equations from (A3.6) are equivalent to these ones.

The sum of the equations (A5.1) and (A5.2) gives $(N_x - 2)w = (V^{11} + V^{22})\overleftarrow{Q}{}^{12}$ and up to (A3.8)

$$w(x) = \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 + \alpha_{22}x_2^2 \tag{A5.5}$$

where α_{ij} are constants.

Let us note that $\alpha_{12}x_1x_2 = 1/2\alpha_{12}(x_1\xi_1 - x_2\xi_2)\overleftarrow{Q}{}^{12}$ and we can regard that $\alpha_{12} = 0$ up to (A3.8).

Substitute (A5.5) to (A5.1), (A5.2), (A5.3) and (A5.4) and obtain

$$
\begin{aligned}
\alpha_{11}x_1^2 - \alpha_{22}x_2^2 &= V^{11}\overleftarrow{Q}{}^{12} \\
-\alpha_{11}x_1^2 + \alpha_{22}x_2^2 &= V^{22}\overleftarrow{Q}{}^{12} \\
2a_{11}x_1x_2 &= V^{12}\overleftarrow{Q}{}^{12} \\
2a_{22}x_1x_2 &= V^{21}\overleftarrow{Q}{}^{12}
\end{aligned}
$$

It follows from these equations that V^{ij} does not contain the terms proportional to $\xi_1\xi_2$. So we can present V^{ij} in the form

$$V^{ij} = v_{ij}^1(x)\xi_1 + v_{ij}^2(x)\xi_2 + v_{ij}^0(x) \tag{A5.6}$$

and obtain

$$
\begin{aligned}
\alpha_{11}x_1^2 - \alpha_{22}x_2^2 &= V^{11}\overleftarrow{Q}{}^{12} = v_{11}^1(x)x_2 - v_{11}^2(x)x_1 & \text{(A5.7)} \\
-\alpha_{11}x_1^2 + \alpha_{22}x_2^2 &= V^{22}\overleftarrow{Q}{}^{12} = v_{22}^1(x)x_2 - v_{22}^2(x)x_1 & \text{(A5.8)} \\
2a_{11}x_1x_2 &= V^{12}\overleftarrow{Q}{}^{12} = v_{12}^1(x)x_2 - v_{12}^2(x)x_1 & \text{(A5.9)} \\
2a_{22}x_1x_2 &= V^{21}\overleftarrow{Q}{}^{12} = v_{21}^1(x)x_2 - v_{21}^2(x)x_1 & \text{(A5.10)}
\end{aligned}
$$

These equations have the following partial solution for v_{ij}^α:

$$
\begin{aligned}
v_{11}^1(x) &= -\alpha_{22}x_2 \ ; \ v_{11}^2(x) = -\alpha_{11}x_1 \\
v_{22}^1(x) &= \alpha_{22}x_2 \ ; \ v_{22}^2(x) = \alpha_{11}x_1 \\
v_{12}^1(x) &= 0 \ ; \ v_{12}^2(x) = -2\alpha_{11}x_2 \\
v_{21}^1(x) &= 2\alpha_{22}x_1 \ ; \ v_{21}^2(x) = 0
\end{aligned}
$$

and the following general solution for $V^{ij}\overleftarrow{Q}{}^{12} = 0$: $v_{ij}^\alpha = x_\alpha v_{ij}(x)$.

So the general solution for V^{ij} and W^{12} obtained from (A3.7) and (A3.6) is

$$w(x) = \alpha_{11}x_1^2 + \alpha_{22}x_2^2 \tag{A5.11}$$

$$
\begin{aligned}
V^{11} &= f_{11}(x)\langle p, p\rangle + g_{11}(x) + (-\alpha_{11}x_1\xi_2 - \alpha_{22}x_2\xi_1) & \text{(A5.12)} \\
V^{22} &= f_{22}(x)\langle p, p\rangle + g_{22}(x) + (\alpha_{11}x_1\xi_2 + \alpha_{22}x_2\xi_1) \\
V^{12} &= f_{12}(x)\langle p, p\rangle + g_{12}(x) - 2\alpha_{11}x_2\xi_2 \\
V^{21} &= f_{21}(x)\langle p, p\rangle + g_{21}(x) + 2\alpha_{22}x_1\xi_1
\end{aligned}
$$

Equation (A3.5): Let us substitute (A5.12) to (A3.5): $V^{i\alpha}\overleftarrow{P}_{j\beta} - V^{j\beta}\overleftarrow{P}_{i\alpha} + V^{j\alpha}\delta_{i\beta} - V^{i\beta}\delta_{j\alpha} = 0$. Consider the case
$i = 1, \ \alpha = 2, \ j = 2, \ \beta = 1$:

$$V^{12}\overleftarrow{P}_{21} - V^{21}\overleftarrow{P}_{12} + V^{22} - V^{11} = 0$$

$x_1\frac{\partial}{\partial x_2}f_{12}(x)\langle p,p\rangle + x_1\frac{\partial}{\partial x_2}g_{12}(x) - 2\alpha_{11}x_1\xi_2 - (-2\alpha_{11}x_2\xi_1) -$
$-x_2\frac{\partial}{\partial x_1}f_{21}(x)\langle p,p\rangle - x_2\frac{\partial}{\partial x_1}g_{21}(x) - 2\alpha_{22}x_2\xi_1 + (2\alpha_{22}x_1\xi_2) +$
$f_{22}(x)\langle p,p\rangle + g_{22}(x) + (\alpha_{11}x_1\xi_2 + \alpha_{22}x_2\xi_1) -$
$-f_{11}(x)\langle p,p\rangle - g_{11}(x) + (\alpha_{11}x_1\xi_2 + \alpha_{22}x_2\xi_1) = 0.$
Extract from the left hand side polynomial the terms $x_2\xi_1$ and $x_1\xi_2$:
$2\alpha_{11}x_2\xi_1 = 0$
$2\alpha_{22}x_1\xi_2 = 0$
So

$$\alpha_{11} = \alpha_{22} = 0$$

Equation (A3.5) again: Now $W^{12} = 0$, $V^{ij} = f_{ij}(x)\langle p,p\rangle + g_{ij}(x)$.
There exist the solutions of the equations

$$V^{ij} = (f(x)\langle p,p\rangle + g(x))\overleftarrow{P}_{ij} = x_j\frac{\partial f(x)}{\partial x_i}\langle p,p\rangle + x_j\frac{\partial g(x)}{\partial x_i} \qquad (A5.13)$$
$$W^{12} = 0$$
$$U^{ij} = -(f(x)\langle p,p\rangle + g(x))\overleftarrow{M}_{ij}$$

Let us look for the solution of (A3.5) up to (A5.13). As the function $\langle p,p\rangle$ commutes with all the operators L^{AB}, one can consider (A3.5) for f_{ij} and g_{ij} separately, namely

$$x_l\partial_k f_{ij} - x_j\partial_i f_{kl} + f_{kj}\delta_{il} - f_{il}\delta_{kj} = 0$$

Consider 3 cases:
1122)
$i = 1, \ j = 1, \ k = 2, \ l = 2$:

$$x_2\partial_2 f_{11} - x_1\partial_1 f_{22} = 0$$

So $\partial_2 f_{11} = x_1\varphi(x), \ \partial_1 f_{22} = x_2\varphi(x)$ and
$f_{11} = \int_0^{x_2} x_1\varphi(x)dx_2 + \varphi_1(x_1),$
$f_{22} = \int_0^{x_1} x_2\varphi(x)dx_1 + \varphi_2(x_2),$
So
$f_{11} = x_1\frac{\partial}{\partial x_1}\left(\int_0^{x_1}\int_0^{x_2}\varphi(x)dx_1dx_2 + N_x^{-1}\varphi_1(x_1) + N_x^{-1}\varphi_2(x_2)\right) + c_{11},$
$f_{22} = x_2\frac{\partial}{\partial x_2}\left(\int_0^{x_1}\int_0^{x_2}\varphi(x)dx_1dx_2 + N_x^{-1}\varphi_1(x_1) + N_x^{-1}\varphi_2(x_2)\right) + c_{22},$
where c_{ii} are constant.
1112) and 2212)
$i = 1, \ j = 1, \ k = 1, \ l = 2$:

$$x_2\partial_1 f_{11} - x_1\partial_1 f_{12} - f_{12} = 0$$

$$i = 2, \; j = 2, \; k = 1, \; l = 2$$

$$x_2 \partial_1 f_{22} - x_2 \partial_2 f_{12} + f_{12} = 0$$

The sum of these equations gives

$N_x f_{12} = x_2 \partial_1 (f_{11} + f_{22}) = x_2 \partial_1 N_x \left(\int_0^{x_1} \int_0^{x_2} \varphi(x) dx_1 dx_2 + N_x^{-1} \varphi_1(x_1) + N_x^{-1} \varphi_2(x_2) \right).$

So $f_{12} = x_2 \partial_1 \left(\int_0^{x_1} \int_0^{x_2} \varphi(x) dx_1 dx_2 + N_x^{-1} \varphi_1(x_1) + N_x^{-1} \varphi_2(x_2) \right) + c_{12}$, where c_{12} is constant.

Thus we obtain that up to (A5.13) the solution of (A3.5) has the form

$$V^{ij} = a_{ij} \langle p, p \rangle + b_{ij}$$

with constant a_{ij}, b_{ij}. Substituting this expression to (A3.5) we finally obtain

$$V^{ij} = a \delta_{ij} \langle p, p \rangle + b \delta_{ij}$$

Equations (A3.2), (A3.3) and (A3.4): When $W^{ij} = 0$ and $V^{ij} = a \delta_{ij} \langle p, p \rangle + b \delta_{ij}$, the equations (A3.2), (A3.3) and (A3.4) take the form

$$U^{ij} \overleftarrow{M}_{kl} + U^{kl} \overleftarrow{M}_{ij} = 0, \tag{A5.14}$$

$$U^{ij} \overleftarrow{P}_{k\alpha} + U^{ik} \delta_{j\alpha} + U^{jk} \delta_{i\alpha} = 0 \tag{A5.15}$$

$$U^{ij} \overleftarrow{Q}_{\alpha\beta} = 0 \tag{A5.16}$$

The solution of (A5.16) has the form

$$U^{ij} = u_{ij}^1(x) \langle p, p \rangle + u_{ij}^0(x) \tag{A5.17}$$

and the equations (A5.14) and (A5.15) can be considered for u^α separately. Eq. (A5.15) gives

$$u_{11}^\alpha \overleftarrow{P}_{22} = 0$$
$$u_{11}^\alpha \overleftarrow{P}_{11} + 2u_{11}^\alpha = 0$$

which implies $(N_x + 2)u_{11}^\alpha = 0$ and as a consequence $u_{11}^\alpha = 0$ (because u_{11}^α is a polynomial). Analogously $u_{22}^\alpha = 0$.

Further, Eq. (A5.15) gives

$$u_{12}^\alpha \overleftarrow{P}_{11} + u_{21}^\alpha = 0$$
$$u_{12}^\alpha \overleftarrow{P}_{22} + u_{12}^\alpha = 0$$

Taking in account the relation $u_{12}^\alpha = u_{21}^\alpha$, the sum of the last 2 equations gives $(N_x + 2)u_{12}^\alpha = 0$ and $u_{12}^\alpha = 0$.

So, up to (A3.8), the solution of the equations (A3.2)-(A3.7) has the form

$$U^{ij} = 0$$
$$V^{ij} = a \delta_{ij} \langle p, p \rangle + b \delta_{ij} \text{ with constant } a \text{ and } b$$
$$W^{ij} = 0$$

Appendix 6. The proof of Proposition A3.1 for $n > 2$.

The case $n > 2$ we prove using induction hypothesis.

Let for $n \leq N - 1$ lemma is true. Consider the case $n = N$. According the inductive hypothesis we can regard, that up (A3.8) the solution has the form

$$
\begin{aligned}
W^{ij} &= 0 \ \text{ for } i, j = 2, ..., N \\
V^{ij} &= \delta_{ij}(v_0(x_1, \xi_1) + v_1(x_1, \xi_1)\langle p, p \rangle) \ \text{ for } i, j = 2, ..., N \\
U^{ij} &= 0 \ \text{ for } i, j = 2, ..., N
\end{aligned}
\tag{A6.1}
$$

Equation (A3.7): The equation (A3.7) gives

$$
W^{1i}\overleftarrow{Q}_{jk} = W^{1i}\left(\frac{\overleftarrow{\partial}}{\partial \xi_j}x_k - \frac{\overleftarrow{\partial}}{\partial \xi_k}x_j\right) = 0 \ \text{ for } i, j, k = 2, ..., N
\tag{A6.2}
$$

It is evident that W^{1i} is at most linear on ξ_s for all ξ_s with $s > 1$ due to (A6.2). Decompose W^{1i}:

$$
W^{1i} = w_0^i(x, \xi_1) + \sum_{s=2}^{K} w_s^i(x, \xi_1)\xi_s
$$

Eq. (A3.7) gives

$$
w_j^i(x, \xi_1)x_k = w_k^i(x, \xi_1)x_j,
$$

which implies

$$
w_j^i(x, \xi_1) = 2w^i(x, \xi_1)x_j,
$$

which implies in its turn

$$
W^{1i} = W_0^i(x, \xi_1) + W_1^i(x, \xi_1)\langle p, p \rangle.
$$

Consider the remaining equations from Eq. (A3.7):

$$
W^{1i}\overleftarrow{Q}_{1k} = \left(W_0^i(x, \xi_1) + W_1^i(x, \xi_1)\langle p, p \rangle\right)\left(\frac{\overleftarrow{\partial}}{\partial \xi_1}x_k - \frac{\overleftarrow{\partial}}{\partial \xi_k}x_1\right) = 0 \ \text{ for } i, k = 2, ..., N
$$

which implies that W_α^i do not depend on ξ_1.

So

$$
W^{1i} = \left(W_0^i(x) + W_1^i(x)\langle p, p \rangle\right) \ \text{ for } i = 2, ..., N
\tag{A6.3}
$$

Equation (A3.6): The equation (A3.6) gives

$$
W^{\alpha\beta}\overleftarrow{P}_{i\gamma} - W^{\alpha\gamma}\delta_{i\beta} + W^{\beta\gamma}\delta_{i\alpha} = V^{i\gamma}\overleftarrow{Q}_{\alpha\beta}
\tag{A6.4}
$$

where $\overleftarrow{Q}_{1\beta} = \frac{\overleftarrow{\partial}}{\partial \xi_1}x_\beta - \frac{\overleftarrow{\partial}}{\partial \xi_\beta}x_1$ $(\beta > 1)$.

The case $\alpha > 1$, $\beta > 1$, $i = 1, \gamma \geq 1$ gives

$$
V^{1i} = \left(V_0^i(x, \xi_1) + V_1^i(x, \xi_1)\langle p, p \rangle\right) \ \text{ for } i = 1, 2, ..., N
\tag{A6.5}
$$

The case $\alpha = 1$, $\beta > 1$, $i = \gamma$ gives

$$(N_x - 2)\, W^{1\beta} = \left(\sum_{i=1}^{N} V^{ii} \right) \overleftarrow{Q}_{1\beta}$$

It follows from (A6.5) that $\left(\sum_{i=1}^{N} V^{ii} \right) \overleftarrow{Q}_{\alpha\beta} = 0$ for $\alpha, \beta > 1$. So, up to (A3.8) we have that $W_0^i(x)$ and $W_1^i(x)$ are polynomials of the second order.

$$W^{1i} = \left(W_0^i(x) + W_1^i(x)\langle p, p \rangle \right) \quad \text{for } i = 2, ..., N \tag{A6.6}$$

Evidently, $W_0^i(x)$ and $W_1^i(x)$ satisfy (A3.6) separately.

The case $\alpha = 1$, $\beta > 1$, $i > 1, \gamma = \beta \neq i$ gives

$$W_k^\beta(x) \frac{\overleftarrow{\partial}}{\partial x_i} x_\beta = 0$$

and so W_k^β depends on x_1 and x_β only.

The case $\alpha = 1$, $\beta = i > 1$, $i \neq \gamma > 1$ gives $W_k^\beta(x)\frac{\overleftarrow{\partial}}{\partial x_\beta} x_\gamma - W_k^\gamma = 0$.

So

$$W_k^\beta = c_k x_1 x_\beta$$

and $W^{\alpha\beta} = (c_0 x_1 \xi_1 + c_1 x_1 \xi_1 \langle p, p \rangle) \overleftarrow{Q}_{\alpha\beta}$. Because $(c_0 x_1 \xi_1 + c_1 x_1 \xi_1 \langle p, p \rangle) \overleftarrow{L}^{AB}$ has the form (A6.1) we can regard that $W^{\alpha\beta} = 0$ up to (A3.8).

With $W^{\alpha\beta} = 0$ the equation (A6.4) gives $V^{ij} \overleftarrow{Q}_{1\beta} = 0$ which implies that

$$V^{ij} = \left(V_0^{ij}(x) + V_1^{ij}(x)\langle p, p \rangle \right) \quad \text{for } i, j = 1, 2, ..., N \tag{A6.7}$$

where $V_k^{ij} = 0$ if $1 < i \neq j > 1$.

So the equation (A3.5) is valid for $V_0^{ij}(x)$ and $V_1^{ij}(x)$ separately.

Equation (A3.5): The equation (A3.5) has the following form for the functions depending on x only:

$$x_\beta \frac{\partial}{\partial x_j} v^{i\alpha} - x_\alpha \frac{\partial}{\partial x_i} v^{j\beta} + v^{j\alpha}\delta_{i\beta} - v^{i\beta}\delta_{j\alpha} = 0$$

Consider the case $i = \alpha = 1$, $j = \beta > 1$. We have $x_j \frac{\partial}{\partial x_j} v^{11} - x_1 \frac{\partial}{\partial x_1} v^{jj} = 0$. Because $v^{jj}(x)$ does not depend on x_j for $j > 1$ we have $v^{jj} = const$ and $\frac{\partial}{\partial x_j} v^{11} = 0$.

Consider the case $i = 1$, $j > 1$, $\beta > 1$, $\beta \neq j = \alpha$. We have $x_\beta \frac{\partial}{\partial x_j} v^{1j} - v^{1\beta} = 0$.

Consider the case $\alpha = 1$, $i = j = \beta > 1$. We have $x_i \frac{\partial}{\partial x_i} v^{i1} + v^{i1} = 0$ which implies $v^{i1} = 0$ because v^{i1} are polynomials.

Consider the case $i = 1$, $\alpha = j = \beta > 1$. We have $x_j(\frac{\partial}{\partial x_j} v^{1j} - \frac{\partial}{\partial x_1} v^{jj}) + v^{jj} = 0$. Because $v^{jj} = const$ this gives $v^{jj} = 0$. So $\frac{\partial}{\partial x_j} v^{1j} = 0$.

Consider the case $i = \alpha = j = 1$, $\beta > 1$. We have $x_\beta \frac{\partial}{\partial x_1} v^{11} - x_1 \frac{\partial}{\partial x_1} v^{1\beta} - v^{1\beta} = 0$ which gives $v^{1\beta} = x_\beta v(x_1)$ and finally $v^{1\beta} = 0$, $v^{11} = const$.

Thus, $V^{ij} = (c_0 + c_1\langle p, p \rangle)\delta_{ij}$.

Equations (A3.2), (A3.3) and (A3.4):

$$U^{ij}\overleftarrow{M}_{kl} + U^{kl}\overleftarrow{M}_{ij} = 0, \tag{A6.8}$$

$$U^{ij}\overleftarrow{P}_{k\alpha} + U^{ik}\delta_{j\alpha} + U^{jk}\delta_{i\alpha} = 0, \tag{A6.9}$$

$$U^{ij}\overleftarrow{Q}_{\alpha\beta} = 0 \tag{A6.10}$$

where $U^{ij} = 0$ for $i, j = 2, ..., N$.

Consider the equation (A6.9) at $k = 1$, $i = j = \alpha > 1$. We have $2U^{1i} = 0$.

Consider the equation (A6.9) at $i = k = 1$, $j = \alpha > 1$. We have $U^{11} = 0$.

Thus, up to (A3.8)

$$
\begin{aligned}
W^{ij} &= 0 \\
V^{ij} &= (c_0 + c_1\langle p, p \rangle)\delta_{ij} \\
U^{ij} &= 0
\end{aligned}
$$

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